OPTIMIZATION IN MULTIVARIATE SAMPLE SURVEY

DISSERTATION

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BY

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Certificate

This is to certify that **Mr. Arvind Kumar** has carried out the work reported in the present dissertation entitled **"optimization in multivariate sample survey"** under my supervision. This dissertation is Mr.Arvind's original work and I recommend it for consideration for the award of Master of Philosophy in Statistics.

(Dr. S.M.Arshad)
Supervisor

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PREFACE

This dissertation entitled **"Optimization in multivariate sample survey"** is submitted to the Aligarh Muslim University, Aligarh for the partial fulfillment of the degree of M.Phil. In this dissertation an attempt has been made to formulate the problem arising during optimization multivariate survey, and to solve them by using the techniques of non-linear programming or otherwise. This manuscript consists of five chapters.

Chapter-1:- deals with the basic ideas of sample surveys and different methods of optimization along with their solution procedures. Programmings with multiple objectives are also presented in this introductory chapter.

Chapter-2:- deals with the problem of optimum allocation in multivariate stratified sampling, and the solution of this problem that have appeared recently are also considered .

Chapter-3:- deals with the problem of allocation when auxiliary information is available in the form of joint distribution of stratification, variable with main variable. The cases where overhead cost is constant and where it is a function of sample number have also been discussed.

Chapter-4:- we describe the use of multivariate auxiliary information through the construction of multivariate ratio and regression estimate and using also multivariate in case of sampling on many occasions.

Chapter-5:- the problem of optimum allocation in sampling with many estimate variables has been formulated in this chapter. A heuristic procedure for its solution is also discussed.

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CHAPTER I

Introduction

1.1 Sample survey:-

In a statistical investigation the interest usually lies in the assessment of general magnitude and the study of variation with respect to one or more characteristics relating to individual belonging to a group. This group of individual under study is called population. Required information in many factors of interest, data are obtained through design and control of statistical experiments or collected and recorded by observation or enquiry. The data about any population can be collected either by census or by sample surveys. A census or complete enumeration is that in which all the elements constituting the population are studied and conclusion are drawn therefrom.On the other hand in sample survey only a small portion of the population is selected. This portion of the population is known as the sample. The population characteristics are constructed on the basis of the result obtained from the sample. In the broadest sense the purpose of sample survey is the collection of information to satisfy a definite need. The need to collect data arises in every conceivable sphere of human activity.

Complete enumeration and sample surveys presuppose the existence of a certain minimum of facilities such as funds, professional personnel for planning the survey methodology and supervision of field operation, sufficiently qualified investigators, sampling frame such as list of units, maps of area units, machine tabulation equipments, transport and communication facilities etc. These facilities or combinations there of do not always exist to the extent needed for a complete enumeration survey and hence in such case it is impossible to have a complete enumeration.

A sample survey is less costly than a complete census because the expenses of covering all units would be greater than that of covering only a sampling fraction. Also it takes less time to collect and process the data from a sample than that of census. The results from a carefully planned and well executed sample survey are expected to be more accurate than those of complete census. A complete census ordinarily requires a huge and unwidely organization and therefore many types of errors creep in, which can not be controlled adequately.

In sample survey the volume of work is reduced considerably and it becomes possible to employ person of higher caliber, trained them suitably and supervise their work effectively. In a sample survey it is possible to make a valid estimate of the margin of error, and hence to decide the accuracy of the result.

Sampling enquiries are becoming more and more popular in all spheres but they are specially advantages in case of social surveys. The large universe (population), difficulties in contacting people, high response etc., make sampling the best procedure in case of social investigation. Recent development in the science of statistics, especially in the field of sampling, have made there procedure more realistic and reliable. In the planning of surveys, the sample will generally involve much fewer respondents, than a census, for which all units in the field are covered are respondents. Practically no one has time or means to make a complete investigation of every problem which he comes in to contact; he must therefore proceed by sample. The main aim and object of the sampling method is to obtain maximum information about the phenomenon under study with the minimum use of money, time and energy.

1.1.1 Random or probability sampling:-

A sampling procedure which satisfies the following properties is termed as random or probability sampling

(i) A set of distinct samples $s_1, s_2, \ldots\ldots\ldots, s_n$ can be defined

(ii)Each possible sample s_i is assigned a known probability of selection n_i.

(iii) The sampling procedure is capable of selecting any of the possible samples s_i with probability n_i.

(iv) The estimate constructed from any specific sample must be unique.

A sampling procedure which does not satisfy the above properties is termed as non – probability sampling

Since no element of probability is involved in non – probability sampling procedure .They are not capable of further development of the sampling theory.

1.1.2 simple random sampling:-

It is the simplest form of sampling in which all possible samples have been provided with equal chance of being selected. In the following, some or well known result of simple random sampling are stated without proof.

Let a simple random sample of size n has been obtain from a population of size N, There will be $^{N}c_{n}$ possible sample.

Let Y_i = the measurement on i^{th} unit of the population or (sample)

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \qquad \text{(The sample mean)}$$

and

$$\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i \qquad \text{(The population means)}$$

It c an be seen that $\bar{y}$ is the unbiased estimate $\bar{Y}$ with its sampling

variance equal to

$$V(\bar{y}) = \left(\frac{1}{n} - \frac{1}{N}\right) S^2$$

where

$$S^2 = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2$$

an unbiased estimate of S^2 is s^2 where

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

Therefore $v(\bar{y})$ will give us an unbiased estimate of $V(\bar{Y})$

1.1.3 Stratified sampling:-

The principle of sampling is based upon a fundamental assumption that the population to be sampled is homogeneous or sometimes the population is not homogeneous. When the population is heterogeneous, the procedure of stratified sampling is used.

In stratified sampling the population is first divided into various strata or group of items possessing similar characteristics and then from each stratum certain items are selected in accordance with random sampling.

The procedure of stratified sampling is intended to give a better cross-section of the population then that of unstratified sampling. It follows that one would expect the precision of the estimated population values or parameters to be higher in stratified than in unstratified sampling. Stratified sampling is also useful in other ways like the selection of sampling unit, the location and enumeration of the selected unit, distribution and supervision of field work and in general, the whole administration of the survey is generally simplified in stratified sampling.

Apart from a number of advantages, the stratified sampling may have following disadvantages:-

(i) In stratified sampling if the stratum wise lists of the units are not available, it may be costly to prepare the same.

(ii) Bias or error may be made in the sample through improper stratification.

(iii) Disproportionate stratification requires weightage and an undue weightage makes the sample unrepresentative.

In the following some important results of stratified random sampling are stated without proof.

Let the population of size N be divided in L strata of size $N_1, N_{2,}$ ...., N_L .the strata are mutually exclusive and $\sum_{i=1}^{n} N_h = N$.

Furthermore let the simple random samples of size $n_1, n_{2,}$ ....n_L respectively have been drawn independently from $1^{st}, 2^{nd}, 3^{rd}$,......... L^{th} stratum.

Let the measurement on the i^{th} unit of h^{th} stratum is Y_{hi}

For h^{th} stratum let

N_h = denote the total no. of units.

n_h = denote the number of units in the sample.

$W_h = \frac{N_h}{N}$ denote the stratum weight .

$\bar{Y}_h = \frac{\sum_{i=1}^{N_h} Y_{hi}}{N_h}$ denote the stratum mean.

$$\bar{y}_h = \frac{\sum_{i=1}^{n_h} y_{hi}}{n_h}$$ denote the sample mean.

$$S_h^2 = \sum_{i=1}^{N_h} \frac{\left(y_{h_i} - \bar{Y}\right)^2}{N_h - 1}$$ denote the stratum mean square.

$$s_h^2 = \sum_{i=1}^{n_h} \frac{\left(y_{hi} - \bar{y}\right)^2}{n_h - 1}$$ denote the sample mean square

The over all population mean.

$$\bar{Y} = \frac{\sum_{h=1}^{L} \sum_{i=1}^{N_h} y_{hi}}{N_h} = \sum_{h=1}^{L} w_h \bar{y}_h$$

If the sampling within each stratum is sample random then,

$$\bar{y}_{st} = \sum_{n=1}^{L} W_h \bar{Y}_h \qquad \text{.... (1.1.1)}$$

is an unbiased estimate of $\bar{Y}$ with sampling variance

$$V\left(\overline{y_{st}}\right) = \sum_{h=1}^{L} \left(\frac{1}{n_h} - \frac{1}{N_h}\right) W_h^2 S_h^2 \qquad \text{.... (1.1.2)}$$

The value of n_h $, h = 1,2,..........L$ are called allocation of sample sizes to various strata. If the over all sample size

$n = \sum_{h=1}^{L} n_h$ is fixed, Neyman gave the following allocation for $n_h, h = 1,2,\ldots\ldots,L$ which minimize the variance (1.1.2) for fixed budget.

$$n_h = \frac{nW_h S_h}{\sum W_h S_h} \qquad \ldots.(1.1.3)$$

is known as Neyman allocation

The variance (1.1.2) under Neyman allocation is

$$V(\bar{y}_{st})_{Ney} = \frac{\left(\sum W_h S_h\right)^2}{n} - \sum \frac{W_h S_h{}^2}{N} \qquad \ldots..(1.1.4)$$

1.2 <u>use of auxiliary information in sample surveys</u>:-

Any variable of known distribution which is highly correlated with the main estimation variable can be used to increase the precision of the estimated such a variable is termed as auxiliary variable. This auxiliary information may be used in several ways. On the basis of such information one may assign the probabilities to various units for being included into the samples.

When only one auxiliary variable is available this purpose is better achieved through stratification of the population under study. The population is stratified by the help of this auxiliary variate. Firstly one determines the stratum boundaries. Dlenius (1957) worked out the boundaries by using the estimation variable in place of auxiliary variable. Block (1958) utilized the auxiliary information for constructing the stratum boundaries, when one character is the subject of the survey.

The auxiliary information is also used in allocating the sample size to various strata in stratified sampling. Neyman gave the formulae for optimum allocation when the auxiliary information on the estimation variable is available from a past experiences or from pilot survey.

The auxiliary information may also be utilized through constructing the ratio and regression estimates in which one eliminates the effects of the variation in the size of the sampling units from the standard error of the estimated character. In chapter-4 we discuss the use of multi-auxiliary information in constructing the ratio and regression estimates.

1.3 sampling with many estimation variables:-

In certain surveys there are several estimation variables. The use of the auxiliary information in sample allocation to the various strata may increase the precision of estimates of some of the characters while those of the rest may decrease beyond the tolerance limits.

A similar situation arises when we use the auxiliary information on one character for determining the strata boundaries in the surveys for several characters. By fixing certain tolerance limits to the precisions of the less important characters we can maximizes the precision for the most important character. The above problem turns out to be a non-linear programming problem.

The solution procedures for the allocation problem and for fixing the strata boundaries in case of many estimation variables have been discussed in chapter-5.

1.4 Methods of optimization.

The problem of optimizing a smooth and well behaved function of several variables can be solved by using techniques of differential calculus. Many optimization problems whose solutions are unattainable by classical methods of calculus are attacked by the methods of Mathematical programming .A mathematical programming problem is concerned with the determination of a minimum or a maximum of a function of several variables which are required to satisfy a number of constraints. Such situations arise in diverse fields including Engg., Operations research, Management Science etc.

The mathematical representation of general programming problem is given as

$$\text{Min. or Max. } Z = f(\underline{x}) \qquad \text{.... (1.4.1)}$$

Subject to

$$g_i(\underline{x}) \ \{\leq, =, \geq\} \ b_i, i = 1,2\ldots., \quad m \qquad \text{.... (1.4.2)}$$

$$\underline{x} \geq \underline{0} \qquad \text{.... (1.4.3)}$$

where $\underline{x}$ is a vector of n components $x_{1,} x_{2,} \ldots\ldots\ldots\ldots, x_n$.

The function (1.4.1) is called objective function. The conditions in (1.4.2) are called constraints and the restrictions in (1.4.3) are called the non-negativity restrictions. The non-negativity restrictions may also be considered to be included in the constraints (1.4.2). The simplest form of the programming problem is a problem in which the functions f and $g_{i,}$ $i = 1,2,\ldots\ldots\ldots,m$ are all linear. Such problem is termed as linear programming problem (LPP).

The other important classes of mathematical programming are: Quadratic programming, integer programming, Geometric programming, stochastic programming and so forth

1.4.1 Quadratic programming:-

A non-linear programming problem having linear constraints and an objective function which is the sum of a linear and a quadratic form is known as a quadratic programming problem. The quadratic programming problem are computationally the least difficult to handle. For this reason quadratic functions and programs are as widely used as the linear functions and programs are as widely used as the linear functions and programs in modeling the optimization problems.

Mathematical model of a quadratic programming problem (QPP) in vector notations can be given as:-

$$\text{Max} \quad f(\underline{x}) = \underline{c}'\underline{x} + \underline{x}'Dx$$

Such that
$$A\underline{x} = \underline{b}$$
$$\underline{x} \geq 0$$

1.4.2 integer programming :-

In integer programming problems some or all variables are constrained to assume non-negative values. This type of problem is of particular importance in business and industry where, quite often discrete nature of variables is involved in many decision making situations.

Mathematically we can write

$$\text{Max.} \qquad Z = \underline{c}\underline{x} + \underline{d}\underline{y}$$

$$\text{Such that} \qquad A\underline{x} + D\underline{y} \le \underline{b}$$

$$\underline{x} \ge 0 \ \& \ \text{Integer}$$

$$\underline{y} \ge 0$$

These types of problems are known as Mixed Integer programming problem (MIPP). If $\underline{y} = \underline{0}$ then it said to be a pure Integer Programming problem (PIPP). If $\underline{x} = \underline{0}$ then the problem is said to be a linear programming problem (LPP). If further the variables x and y are restricted to take only values zero or one, then the above problem is termed as a zero one Integer programming problem .

1.4.3 Geometric programming:-

In Geometric programming the functions involved are posynomials. Geometric programming derives its names from its relationship with certain geometric concepts. It provides a systematic methods for formulating and solving the class of optimization problems that tend to appear mainly in engineering design. This optimization procedures was largely developed by C.Zener, R.J.Duffin and E.L.Paterson in the early 1960.

In an engineering design the total cost G is a sum of component costs u_g .thus,

$$G = u_1 + u_2 + \dots\dots \dots + u_n .$$

Generally, the component costs are expressed as

$$U_j(x) = c_j x_1^{a_{1j}} x_2^{a_{2j}} \dots\dots .x_m^{a_{mj}}$$

Where $c_j > 0$ $x_i(i = 1,\dots\dots\dots, m) > 0$

And $a_{ij}(i = 1,2,\dots\dots, m : j = 1,2,\dots\dots, n)$ are unrestricted in sign.

The function G is usually referred as posynomial .

1.4.4 <u>Stochastic programming:</u> -

In many cases of practical importance, it turns out that some of the parameters appearing in the problem must be treated as random variables rather than as deterministic ones. We shall refer to the programming problem in which some of the parameters are random variables as stochastic programming problems.

1.5 Solution procedures:-

The usual method for solving a programming problem is to obtain a starting solution which satisfies the constraints and restrictions. Such a solution is called feasible solution. A feasible solution which optimizes the objective function is known as an optimal solution. Before starting any iteration one must check a carefully designed optimality criterion to ascertain that the present solution is optimal or not. No single method is available which is universally applicable to every type of programming problem. However special algorithms are available for almost all classes of programming Problems. Some of them mentioned in the following

Simplex method was devised by G.B.Dantzig to solve linear programming in 1947. The method also indicates whether or not the program is feasible. If the program is feasible, it either finds an optimal solution or indicates that an unbounded solution exists.

Various methods for solving quadratic programming problems are:-

Wolfe (1959), Beale (1959) and Houthakkar (1960). Rosen (1960, 1961) gave his gradient projection method for solving a convex linear programming problem.

In geometric programming the functions involved are posynomials. It provides a systematic method for formulating and solving the class of optimization problem that mainly appear in engineering designs. This procedure was largely developed by C.Zener, R.J.Duffion and E.L.Pattersan in the early 1960s.

In stochastic linear programming problem some of the parameters are random. Stochastic linear programming problems are generally attacked by the two methods namely:-

(1). Two stage programming technique and

(2) Chance constrained programming technique.

Two stage programming technique is one which converts a stochastic linear programming problem in to an equivalent deterministic problem and the technique was suggested by G. B. Dantzig.

A stochastic linear programming problem can be stated as follows:-

$$\text{Min } f(x) = c^T x = \sum_{j=1}^{n} c_j x_j \qquad \dots (1.5.1)$$

$$\text{S.T. } A_i^T x = \sum_{j=1}^{n} a_{ij} x_j \geq b_i, i=1,\dots, m \qquad \dots. (1.5.2)$$

$$x_i \geq 0, \qquad j=1, 2,\dots, n \qquad \dots. (1.5.3)$$

For simplicity, we assume that only the elements b_i are probabilistic. This means that the variable b_i is not precisely known, but its probability distribution function, with a finite mean $\bar{b}_i$ is known to us. In this case, it is impossible to find a vector X in such a way that $A_i^T x$ will be greater than or equal to $b_i (i = 1, \ldots\ldots, m)$ for whatever value b_i takes. In fact the difference between $A_i^T x$ and b_i will itself be a random variable, whose probability distribution function depends on the value of X chosen. The two stage problem is interpreted as follows:

First stage:- first estimate or guess the vector b, and find the vector x by solving the problem stated in equations (1.5.1) to (1.5.3).

Second stage: - then observe the value of b, and hence its discrepancy from the previous guess vector and find the vector $Y = Y(b, X)$ by solving the second stage problem.

The chance constrained programming technique is one which can be used to solve problems involving chance constraints, that are constraints having finite probability of being violated. This chance constrained programming permits the constraints to be violated by a specified (small) amount, whereas the two stages programming does not permit any constraint to be violated. The chance constrained programming technique was originally developed by Charnes and Cooper.

For chance constrained programming, the problem is stated as follows:-

$$\text{Min } f(x) = \sum_{j=1}^{n} c_j x_j \qquad \ldots\ldots(1.5.4)$$

S.T. $$P\left[\sum_{j=1}^{n} a_{ij}x_j \le b_i\right] \ge p_i, i = 1,2,\ldots\ldots,m \qquad \ldots\ldots (1.5.5)$$

And $$x_j \ge 0, \qquad j = 1,2,\ldots\ldots\ldots,n \qquad \ldots\ldots (1.5.6)$$

Where c_j, a_{ij} and b_i are random variable and p_i are specified probabilities. Equation (1.5.5) indicates that the i^{th} constraint,

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i,$$

has to be satisfied with a probability of at least p_i, where $0 \le p_i \le 1$.

1.6 Programming with multiple objective :-

The fact that real life problems occurs with conflicting objectives rather than with a single one has been recognized by the early practitioners of mathematical programming . For example Reinfeld and Vogel (1958) provide a vivid description of the conflicting objective that arises in a manufacturing plant. For the solution of some of these problems they used singles objective linear programming in a model that is known as 'Goal Programming'. Still the development of special techniques to assist management in their quest to deal effectively with multiple objectives came some what later than the importance of the problem would suggest.

A multiple objective linear programming model with n decision variables, 'm' constraints and 'p' objective functions can be stated as follows:

Max $\quad Z = \left[z_1, z_2, \ldots\ldots\ldots, z_p\right]$

$$\text{With} \quad \begin{aligned} z_1 &= z_1(\underline{x}) \\ &\cdot \\ z_p &= z_p(\underline{x}) \end{aligned}$$

Subject to $g_i(x) \leq b_i$, $\quad i=1,2,\ldots\ldots\ldots,m$

And $\quad x_j \geq 0, \quad j=1,2,\ldots\ldots\ldots,n.$

Where $z_p(x_j)$ and $g_i(x_j)$ are linear functions of the decision variables $x_{j,}$ and $b_i\left(b_i \geq 0\right)$ are constant values.

The ideal solution for a multiple objective linear programming problem would be to find that feasible set of decision variables $x_j (j = 1,2,\ldots\ldots,m)$ which would maximize the individual objectives functions of the problem simultaneously. However with conflicting objective in the model, a feasible solution that optimizes one objective function may not optimize any of the others. This means that what is optimal in terms of one of the 'p'objective is generally, not optimal for the other '$p-1$' objectives.

<u>Solution procedures</u>:-

Let p objective function be arranged in decreasing order of priority. The exact implication of the ordering chosen will become clear below.

For k^{th} priority objective we write $\sum_j c_{xj} x_j$ and c_{ko} for its goal.

To start, we consider the first highest priority objective and try to find a feasible solution $(x_1, \ldots, x_n)$, satisfying $\sum_j c_{ij} x_j \leq c_{lo}$ And, if our search is successful, we impose this inequality as an extra constraints and then turn to the second objective. If no such solution can be found, we impose the constraints that the first objective function, should not drop blow its optimal value before turning to the second objective.

If $(\hat{x}_1, \ldots\ldots\ldots, \hat{x}_n)$ is optimal for the first objective function, this implies that, if $\sum_j c_{1j} x_j < c_{10}$, then we impose the constraint

$$\sum_j c_{ij} x_j \geq \sum_j c_{ij} \hat{x}_j \qquad \ldots. (1.6.1)$$

Both cases can be covered by imposing the additional constraint

$$\sum_{j=1} c_{ij} x_j \geq \min\left[c_{1o}, \sum_j c_{ij} x_j \right] \qquad \ldots. (1.6.2)$$

and then turning to second objective. A similar procedure which can be adopted for second objective by imposing additional constraint on the k^{th} objective before proceeding to the $(k+1)^{th}$ is that

$$\sum_{j} c_{kj} x_j - c_{ko} \geq \min \left[o, \sum_{j} c_{kj} x_j - c_{ko} \right] \qquad \text{.... (1.6.3)}$$

in the above procedure we sequential optimize the criteria, starting with the highest priority objective and imposing (1.6.3),which say that we do not permit any reduction in the k^{th} criteria, when passing from the k^{th} to $(k+1)^{th}$ criterion. More generally, we can use weighting factors and priorities. However, deviation variables always allow us to reformulate the problem in the form outlined above, which refer to as a 'Priority Goal Programming problem'.

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CHAPTER II

Optimum Allocation in Multivariate Stratified Sampling

2.1 Introduction:

In stratified random sampling where more than one characters are to be estimated on each unit of the population understudy. No simple procedure is available for obtaining optimum allocations because there is no single optimality criterion through which the problem of stratification can be attacked. The problem discussed here is the problem of selecting the number of units to be sampled from each stratum, Such that the total cost of the survey is minimized under certain restriction imposed on the variance of the different characteristics according to their significance.

A procedure limited to only two strata kokan (1963) discussed this situation as a problem of Non-linear programming and proposed a solution. In this chapter an analytical solution of the multivariate allocation problem presented by Kokan and Khan (1967) has been discussed.

2.2 Optimum allocation of sample size in stratified random sampling.

Stratified random sampling has an important place in the theory of sampling. In stratified sampling the total population $u = u_1, u_2, ... u_N$ is the first partition in to several sub population. These sub population is known as the strata. Population characteristics can be inferred with samples from each stratum, exploiting the gain in precision in the estimates, administrative convenience and flexibility of using different sampling procedures in the different subpopulations.

Let N_i be the number of units in the i^{th} stratum and $\sum_{i=1}^{L} N_i = N$, where L is the number of strata into which the N units are divided. Let 'n_i' be the size of the sample drawn from i^{th} stratum. Assume that the samples are drawn independently in different strata.

The problem of optimality choosing the n_i's is known as the "optimal allocation problem". The objective in this problem might be minimization of variance of the estimate of the population characteristics under study, with restriction on the total number of samples drawn or on the total budget available. Also the objective might be minimization of the total cost of sampling for a desired precision.

First we consider an unbiased estimate of the population mean $\bar{Y}$, Where Y is the characteristics under study. Let $\bar{y}_i$ be an unbiased estimate of the stratum $\overline{Y}_i$ that is, then $\bar{y}_{st}$ given by

$$\bar{y}_{st} = \frac{1}{N} \sum_{i=1}^{L} N_i \bar{y}_i \qquad .....(2.2.1)$$

is an unbiased estimate of the population mean $\bar{Y}$. As the precision of this estimate is measured by the variance of the sample estimate is measured by the variance of the sample estimate, we consider next the variance of $\bar{y}_{st}$' denoted by $V(\bar{y}_{st})$.

$$V(\bar{y}_{st}) = \sum_{i=1}^{L} w_i^2 s_i^2 \left(\frac{1}{n_i} - \frac{1}{N_i} \right)$$

$$= \sum_{i=1}^{L} W_i^2 S_i^2 X_i \qquad \text{........ (2.2.2)}$$

Where $X_i = \dfrac{1}{n_i} - \dfrac{1}{N_i}$

Problem – A

Here we consider the problem of choosing n_i,i=1,2,…,L, such that the sum of these n_i equals n, a fixed total sample size, and the $V(\bar{y}_{st})$ is minimum. This problem can be formulated as

$$\text{Minimize} \sum_{i=1}^{L} W_i^2 S_i^2 X_i$$

$$\text{Subject to} \sum_{i=1}^{L} n_i = n \qquad \text{...... (2.2.3)}$$

$$N_i \geq n_i \geq 1, \ n_i \text{ is integer, } i = 1, 2, \ldots, L,$$

Let $a_i = W_i^2 S_i^2$, i = 1, 2, …. L. Then the objective function

$$\sum_{i=1}^{L} W_i^2 S_i^2 X_i = \sum_{i=1}^{L} a_i X_i$$

$$= \sum_{i=1}^{L} a_i \left(\frac{1}{n_i} - \frac{1}{N_i} \right)$$

$$= \sum_{i=1}^{L} \frac{a_i}{n_i} - \sum_{i=1}^{L} \frac{a_i}{N_i}$$

But $\sum_{i=1}^{L} \frac{a_i}{N_i}$ is a constant. Therefore, it is sufficient to consider minimizing $\sum_{i=1}^{L} \frac{a_i}{n_i}$.

Thus problem A becomes:

$$\text{Minimize } \sum_{i=1}^{L} \frac{a_i}{n_i} \qquad \text{.... (2.2.4)}$$

$$\text{Subject to } \sum_{i=1}^{L} n_i = n$$

$N_i \geq n_i \geq 1, n_i$ integer i=1, 2, ..., L

If the restrictions that n_i must be a positive integer and bounded above by N_i for all i are relaxed, and then the classical Lagrangian Multiplier Method can be used to find optimal n_i.

We have

$$n_i = n \frac{\sqrt{a_i}}{\sum_{i=1}^{L} \sqrt{a_i}} \qquad \text{.....(2.2.5)}$$

However, there are three eventualities:

(i) $n_i > N_i$ for some i or

(ii) n_i may not be an integer for every i, or

(iii) $n_i < 1$ for some i.

In that case, we do not have a solution to problem (2.2.4)

In the sampling literature, eventualities (i) is referred to as over sampling that is, the optimal allocation requires sampling more than 100% in certain strata.

Non-integer solutions are rounded off. Eventuality (iii) can be easily taken care of by assuming that we sample at least one unit from each stratum, and allocating the rest of n-L units optimally. By noticing that $1/n_i$ is strictly convex in each i, we find the objective function to be a strictly convex function if $a_i > o$, that is, $S_i^2 > o$ for all i. Then we are interested in minimizing a strictly convex function over a bounded convex region, created by a linear equality and upper and lower bound restrictions. When L = 2, feasible region and the objective function appear as in fig. 2.1. In fig. 2.1. both N_1 and N_2 are larger than n. otherwise, we may have the configuration shown in fig. 2.2.

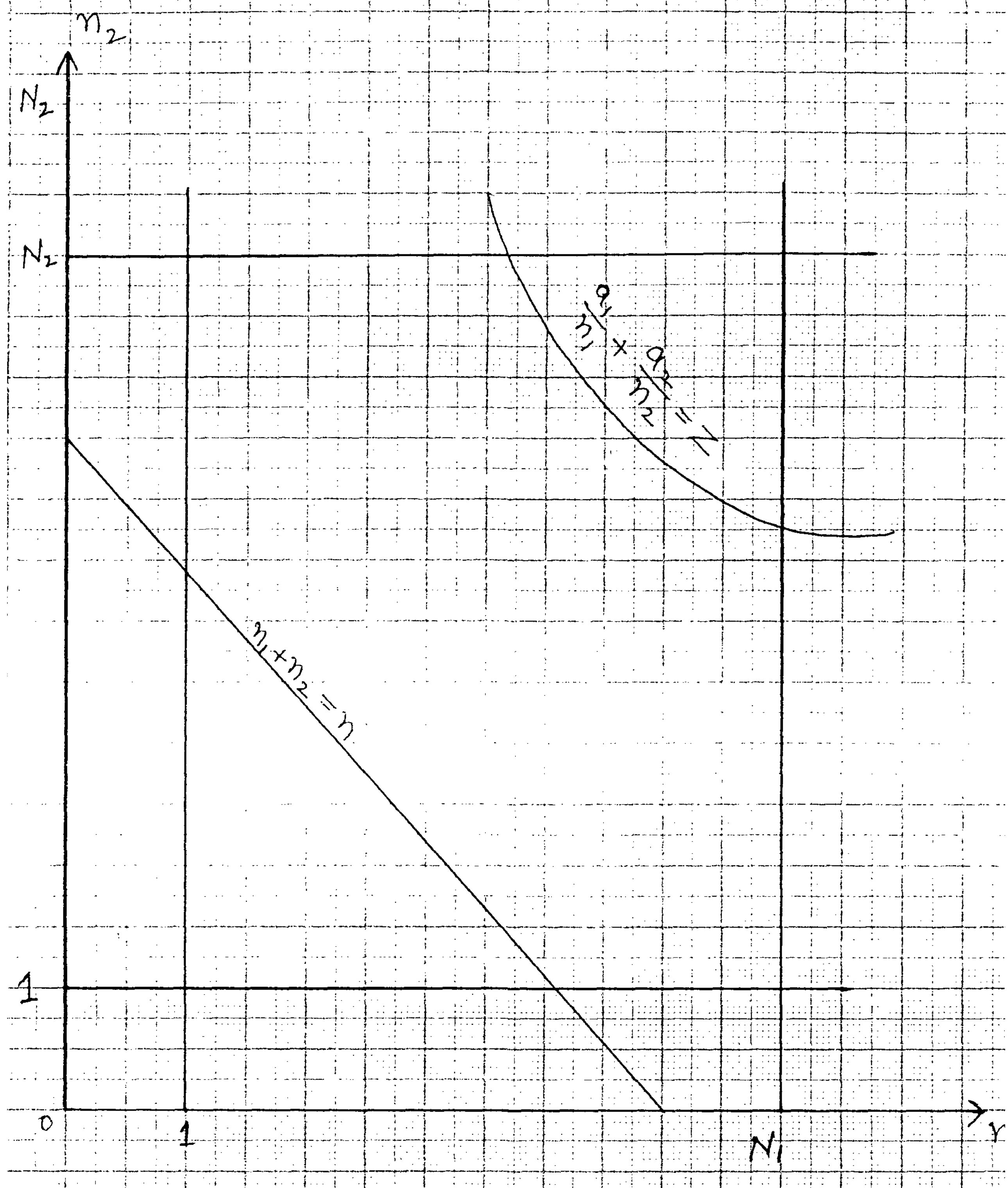
<u>Problem – B</u>

We can also treat similarly the problem of minimizing the total cost of sampling, subject to certain restrictions on the allowable loss in precision. We have the problem stated as follows:

Minimize $\sum_{i=1}^{L} c_i n_i$

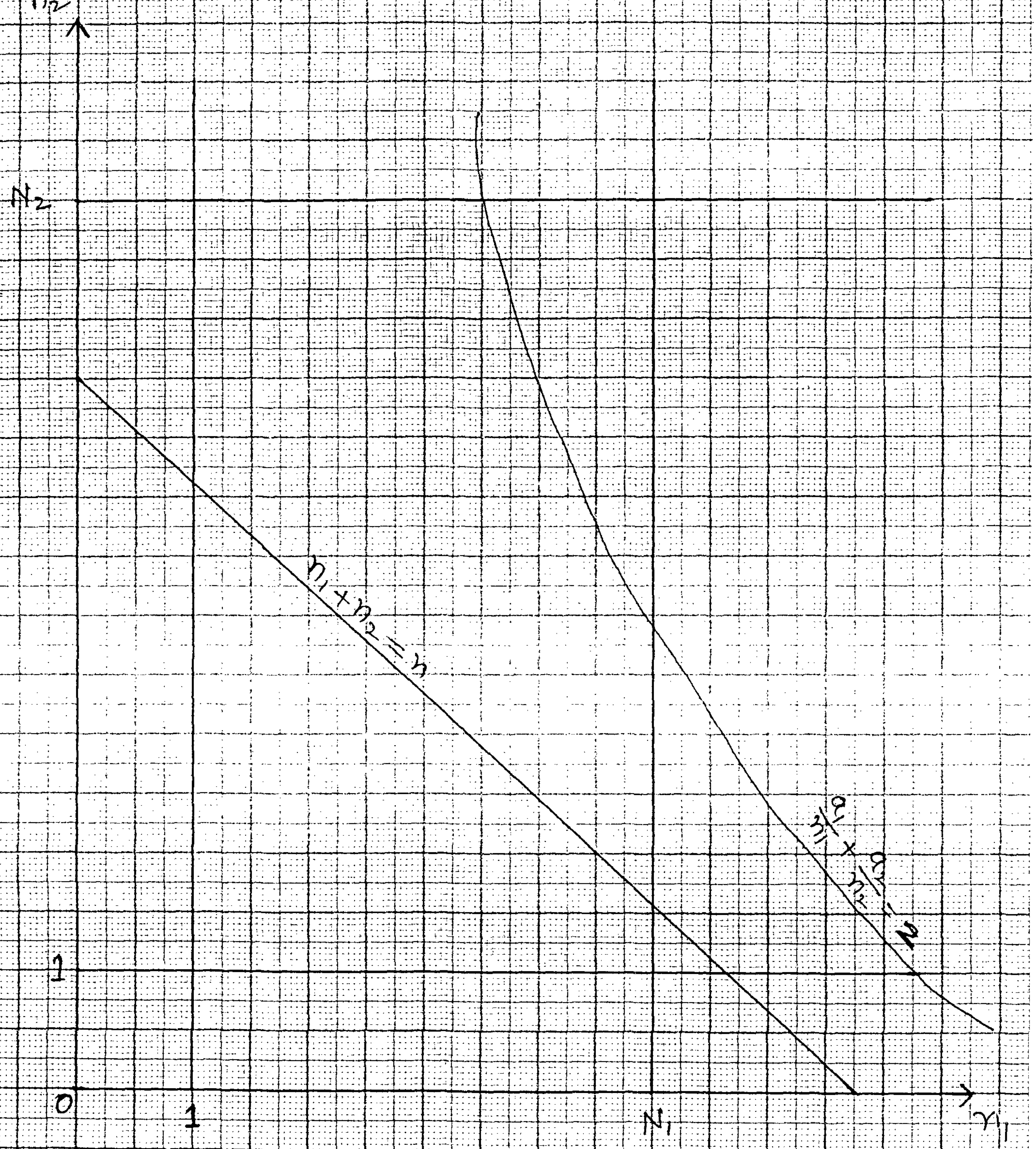
Subject to $\sum a_i / n_i \leq v$ ......(2.2.6)

$1 < n_i < N_i$, n_i integer for i=1, ...,L.



feasible resion and objective function When N_1 and N_2 are both larger then n.

fig (2.1)



feasible region and objective function when only N_2 is larger then n.

fig (2.2)

So far we have considered only one characteristic for study. But if we have conduct multivariate survey i.e. we wish to study several characteristics, the problem of optimal allocation does not yield to such a simple approach. In the next section we consider the problem of minimizing the total cost so as to achieve prescribed precision of the estimates of several populations characteristics.

2.3 Optimum Allocation of sample sizes in Multivariate stratified Random sample:

we have assume there are p characteristics under study. Let Y_j be the j^{th} characteristic considered, as earlier. We have L strata, and N_i units in the i^{th} stratum.

$$\sum_{i=1}^{L} N_i = N$$

Assume that the n_i samples are drawn independently from each stratum. Also assume that $\bar{y}_{ij}$ is an unbiased estimate of $\bar{Y}_{ij}$, that is

$$\bar{y}_{ij} = \frac{1}{n} \sum_{n=1}^{n_i} y_{ijh}$$

Where y_{ijh} is value observed for Y_j in the i^{th} stratum for the h^{th} sample unit. An unbiased estimate of the population mean $\bar{Y}_j$ is given by

$$\bar{y}_j(st) = \frac{1}{N} \sum_{i=1}^{L} N_i \bar{y}_{ij} \qquad .... (2.3.1)$$

Where y_{ijh} is the value observed for Y_j in the i^{th} stratum for the h^{th} sample unit. An unbiased estimate of population characteristics, for each characteristics. As noted in the previous section,

$$v_j = V\left(\bar{y}_j(st)\right) = \sum_{i=1}^{L} W_i^2 S_{ij}^2 X_i \qquad .... (2.3.2)$$

Where

$$W_i = N_i/N, \qquad S_{ij}^2 = \frac{1}{N_i - 1} \sum_{h=1}^{N_i} \left(y_{ijh} - \bar{Y}_{ij}\right)^2$$

and

$$X_i = \frac{1}{n_i} - \frac{1}{N_i}.$$

Let $a_{ij} = W_i^2 S_{ij}^2$. Let c_i be the cost of sampling all the P characteristics on a single unit in the i^{th} stratum. The total variable cost of the survey is

$$K = \sum_{i=1}^{L} c_i n_i$$

Assume a_{ij}, $c_i > o$, for $i = 1, \ldots, L$, $j = 1, \ldots, P$.

The problem of allocation can now be stated as problem C.

Problem C

$$\text{Minimize} \quad \sum_{i=1}^{L} c_i n_i \qquad .... (2.3.3)$$

$$\text{Subject to } \sum_{i=1}^{L} a_{ij} X_i \le v_j \qquad j = 1,\dots\dots,p \qquad \text{.... (2.3.4)}$$

$$o \le X_i \le 1 - \frac{1}{N_i}, \qquad i = 1,\dots\dots, L \qquad \text{.... (2.3.5)}$$

$$X_i = \frac{1}{n_i} - \frac{1}{N_i}, \quad n_i \text{ integer}, i = 1,\dots\dots L \qquad \text{.... (2.3.6)}$$

Where V_j is allowable error in the estimate of the j^{th} characteristic.

Problem C is an integer linear programming problem but for the restriction (2.3.6) which is non-linear. When the new variable $X_i = \frac{1}{n_i}$, $i = 1,\dots L$ is introduced, Problem C can be equivalently stated as problem D.

Problem –D

$$\text{Minimize } \sum_{i=1}^{L} c_i / X_i \qquad \text{.... (2.3.7)}$$

$$\text{Subject to } \sum_{i=1}^{L} a_{ij} X_i \le b_j, \quad j = 1,\dots\dots P \qquad \text{.... (2.3.8)}$$

$$\frac{1}{N_i} \le X_i \le 1, \qquad i = 1,\dots\dots L \qquad \text{.... (2.3.9)}$$

Where $\qquad b_j = v_j + \sum_{i=1}^{L} a_{ij}/N_i, \quad j = 1,\dots\dots P.$

Remark 2.3.1

The objective function (2.3.7) in problem D is a strictly convex function, because c_i/x_i is strictly convex for $c_i > o$.

Remark 2.3.2

The restrictions (2.3.8) and (2.3.9) provided a bounded convex feasible region for the problem, formed by linear inequalities. The region is non- empty as

$$X = \left(\frac{1}{N_i}, \frac{1}{N_2}, \ldots\ldots, \frac{1}{N_L} \right)$$

is feasible. Thus an optimum $X = (X^*1, \ldots\ldots, X^*L)$ exists. Strict convexity also implies uniqueness of the optimal solution.

Remark. 2.3.3

The optimum is attained at a boundary of the convex set. Problem D is a convex programming problem like the type discussed in section 2.2. There we developed the necessary and sufficient condition for a X to be optimal. There are several methods for solving such problems, the convex-simplex method, feasible direction method, gradient projection method, cutting plane method, and so on.

However, all these methods find an X which may correspond to a non-integer n_i, i=1,....,P. Rounding off yields in those cases a near optimal solution. But if we wish to find integer optimal solutions to problem C, we have to resort to some Branch and bound scheme in which several problems of the type of problem D may have to be solved, for the calculating the bounds.

Remark 2.3.4

The optimal solution to problem D provides a lower bound on the value of the optimal solution to problem C. On the other hand, a rounded off integer solution that is feasible for problem D turns out to be an upper bound on the optimal objective function value to problem C. Thus the deviation from the optimum to problem C can be measured, before we go to the branch and bound procedure. Also these bounded can help in terminating the branch and bound procedure at an intermediate stages, as soon as the upper and lower bounds are sufficiently close, for all practical purposes, as too much computer storage and time are required for problems with a large number of variables.

2.4 Geometric Interpretation of the Problem:

We consider the case when L (number of strata) =2

the objective function.

$$Z = c_i/X_1 + c_2/X_2$$

Is equivalent to

$$Z = \frac{c_1 x_2 + c_2 x_1}{x_1 x_2}$$

From this

$$X_1 X_2 = \frac{c_1 x_2}{z} + \frac{c_2 x_1}{z}$$

Or

$$X_1 X_2 - \frac{c_1 x_2}{z} - \frac{c_2 x_1}{z} = 0$$

This yields the equivalent form for the objective function in terms of x_1, x_2 and z as

$$(X_1 - c_1/z)(X_2 - c_2/z) = \frac{c_1 c_2}{z} \qquad \text{.... (2.4.1)}$$

Which is a rectangular hyperbola with center (c_1/z, c_2/z). As z varies, the center (c_1/z, c_2/z) lies on the line x_2/x_1

$$\frac{x_2}{x_1} = \frac{c_2}{c_1} \qquad \text{.... (2.4.2)}$$

And the vertex of the rectangular hyperbola

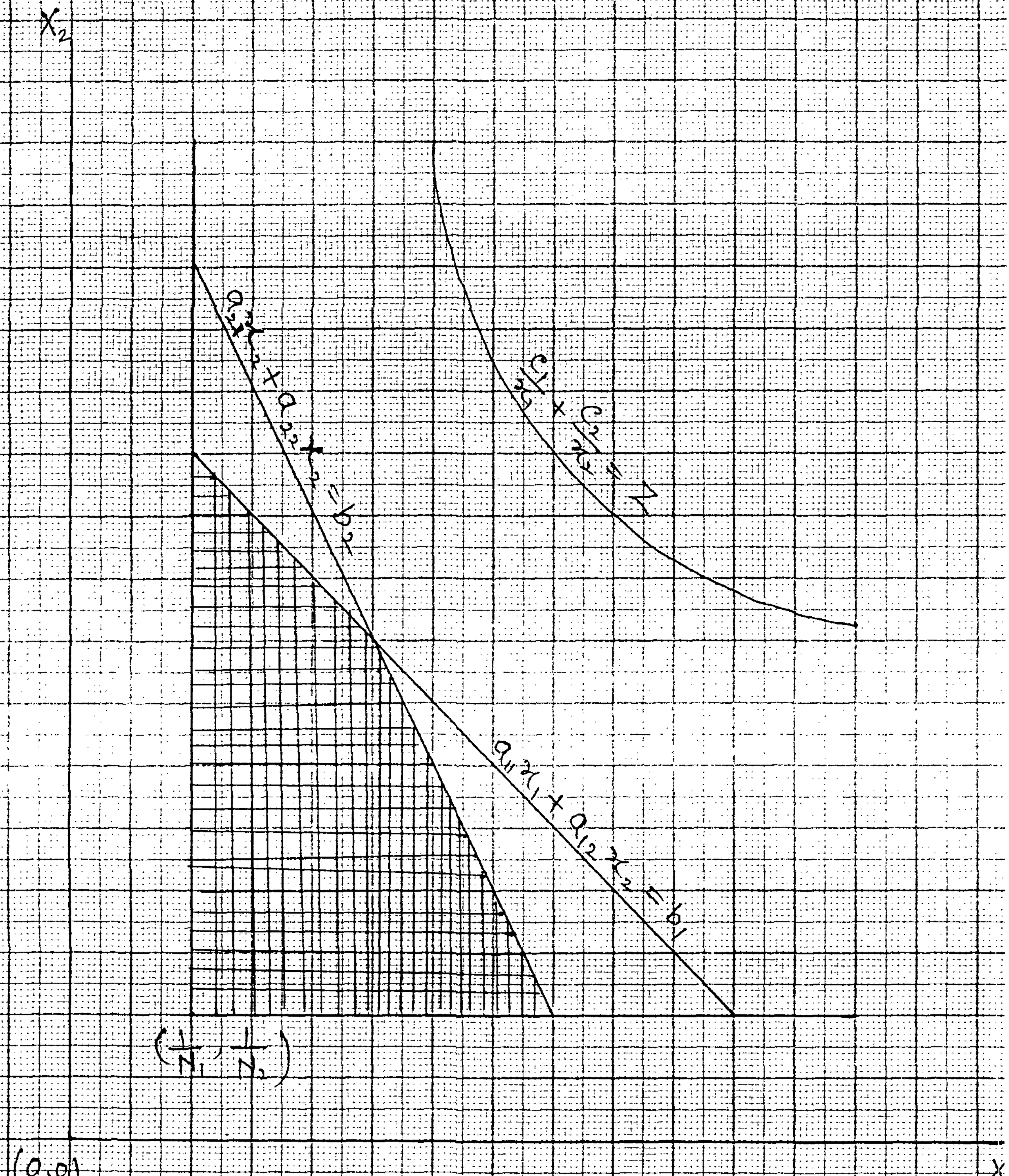
$\left(\left|c_1 + \sqrt{c_1 c_2}\right|/z, \left|c_2 + \sqrt{c_1 c_2}\right|/z\right)$ lies on the line

$$x_2/x_1 = \frac{\left|c_2 + \sqrt{c_1 c_2}\right|}{c_1 + \sqrt{c_1 c_2}} \qquad \text{.... (2.4.3)}$$

Now consider the restrictions of (2.3.8) and (2.3.9). We have the feasible region in the non-negative orthand, as $a_{1j}x_1 + a_{2j}x_2 = b_j$ has negative slope and positive x_2-intercepts in the x_1x_2-plane, and the upper and lower bounds on x_1 and x_2 are positive.

To obtain the optimum allocation we have to find the rectangular hyperbola (2.4.1) for some value of z such that it touches the boundary of the feasible region. See fig.2.3.

In general when we have L strata we have the following results.



Objective function and the feasible region

fig (3.1)

Result 2.4.1

The point of contact of the hyper plane

$\sum a_i x_i = b \qquad (a_i, b>0)$

with the objective function

$$Z = \sum_{i=1}^{L} \frac{c_i}{x_i}$$

Is given by $x = (x_1, \ldots\ldots\ldots, x_L)$, where

$$x_i = \frac{\left| b\sqrt{c_i a_i} \right|}{\left| a_i \sum_{i=1}^{L} \sqrt{c_1 a_i} \right|}, \quad i = 1, \ldots., L \qquad \ldots. (2.4.4)$$

Proof: The objective function can be written as

$$f(x_1, \ldots, x_L) = \sum_{i=1}^{L} c_i \prod_{\substack{h=1 \\ h \neq 1}}^{L} x_h - z \prod_{h=1}^{L} x_h \qquad \ldots.(2.4.5)$$

Let $f_x(x^1)$ denote $(\partial F/\partial x_1, \ldots\ldots \partial F/\partial X_L)$ evaluated at $x^{(1)}$, Let $f_x(x^1)_i$ denote the i^{th} coordinate of $f_x(x^1)$. Then

$$f_x(x^1)_i = \sum_{\substack{k=1 \\ i \neq k}}^{L} c_k \prod_{\substack{h=1 \\ k \neq h \neq 1}}^{L} x_h^1 - z \prod_{\substack{h=1 \\ h \neq i}}^{L} x_h^{(1)}$$

Thus we have the equation for hyper plane touching the objective function at $x^{(1)}$, given by

$$\sum_{i=1}^{L}\left(x_i - x_i^{(1)}\right) f_x\left(x^1\right)_i = 0 \qquad \text{... (2.4.6)}$$

Or

$$\sum_{i=1}^{L} x_i \left[\sum_{\substack{k=1 \\ i\neq k}}^{L} c_k \prod_{\substack{h=1 \\ k\neq h\neq i}}^{L} x_h^{(1)} - z \prod_{\substack{h=1 \\ h\neq i}}^{L} x_h^{(1)}\right]$$

$$+\left\{\left(-\sum_{i=1}^{L} x_i^{(1)}\right)\left[\sum_{\substack{h=1 \\ i\neq k}}^{L} c_k \prod_{\substack{h=1 \\ k\neq h\neq i}}^{L} x_h^{(1)}\right] + z \prod_{\substack{h=1 \\ h\neq i}}^{L} x_h^{(1)}\right\}$$

Since the term in the brackets{ } is equal to $z \prod_{h=1}^{L} x_h^{(1)}$

After simplification we have

$$\sum_{i=1}^{L} x_i \left[\sum_{\substack{k=1 \\ i\neq k}}^{L} c_k \prod_{\substack{h=1 \\ k\neq h\neq i}}^{L} x_h^{(1)} - z \prod_{\substack{h=1 \\ h\neq i}}^{L} x_h^{(1)}\right] + z \prod_{h=1}^{L} x_h^{(1)} = 0 \qquad \text{....(2.4.7)}$$

This hyper plane will represent the hyper plane

$\sum_{i=1}^{L} a_i x_i = b$ in case

$$\left[\sum_{\substack{k=1\\k\neq i}}^{L} c_k \prod_{\substack{h=1\\k\neq h\neq i}}^{L} x_h^{(1)} - z \prod_{\substack{h=1\\h\neq i}}^{L} x_h^{(1)}\right] / a_i = -z \prod_{h=1}^{L} x_h^{(1)} / b \qquad \ldots. (2.4.8)$$

The implication is that

$$\frac{1}{a_i} \sum_{\substack{k=1\\i\neq k}}^{L} c_k \prod_{\substack{h=1\\k\neq h\neq i}}^{L} x_h^{(1)} = z \prod_{\substack{h=1\\h\neq i}}^{L} x_h^{(1)} \left[\frac{1}{a_i} - \frac{x_i^{(1)}}{b}\right] \qquad \ldots.. (2.4.9)$$

Dividing both sides of (2.4.9) by $\prod_{\substack{h=1\\h\neq i}}^{L} x_h^{(1)}$, we get

$$\frac{1}{a_i} \sum_{\substack{k=1\\i\neq k}}^{L} \frac{c_k}{xk^{(1)}} = z \frac{b - a_i x_i^{(1)}}{a_i b_j} \qquad \ldots.. (2.4.10)$$

Canceling out $1/a_i$ and adding and subtracting

$c_i / x_i^{(1)}$ In the left hand side of (2.4.10) we get

$$\sum_{k=1}^{L} \frac{c_k}{x_k^{(1)}} - \frac{c_i}{x_i^{(1)}} = z \left[\frac{b - a_i x_i^{(1)}}{b}\right]$$

But

$$\sum_{k=1}^{L} \frac{c_k}{x_k^{(1)}} = z \,.$$

Hence, after substitution and simplification, we get

$$x_i^{(1)} = \sqrt{\frac{c_i b}{a_i z}}, \qquad i=1,\dots L \qquad \text{..... (2.4.11)}$$

Now

$$z = \sum \frac{c_i}{x_i^{(1)}} = \sum \frac{c_i}{\sqrt{\frac{c_i b}{a_i z}}}$$

The implication is that

$$\sqrt{z} = \frac{1}{\sqrt{b}} \sum_{i=1}^{L} \sqrt{c_i a_i} \qquad \text{..... (2.4.12)}$$

Eliminating the z in expression (2.4.11) we finally obtain.

$$x_i^{(1)} = \frac{b\sqrt{c_i a_i}}{a_i \sum_{i=1}^{L} \sqrt{c_i a_i}}, \qquad i=1,\dots\dots,L$$

As required. Introducing the subscript 'j' for the different characteristics, we have the corresponding result for the jth hyper plane.

We now can describe a procedure which is efficient in case for a certain 'j', the $x_{ij}^{(1)}$ discussed in Result 2.4.1.for the characteristics j satisfies all the constraints

$$\sum a_{ij} x_{ij}^{(1)} \le b_j, \qquad j = 1,....,P$$

and

$$\frac{1}{N_i} \le x_{ij}^{(1)} \le 1, \qquad i = 1,....,L$$

Step.1. We discard from the set of constraints (2.3.8) those which are not binding i.e., we find the intercepts $b_j/a_{ij}, ..., b_j/a_{ij}$ for each j and discard those, j for which the vector of intercepts strictly dominates the corresponding vector for any other j. Assume that i_1 is the set of binding constraints among the constraints (2.3.8).

Step 2. Compute $x_j = (x_{ij},....,x_{Lj})$ for each characteristics $j \in I_1$, using result 2.4.1, that is

$$x_{ij} = \frac{b_j \sqrt{c_i a_{ij}}}{a_{ij} \sum_{i=1}^{L} \sqrt{c_i a_{ij}}}$$

Step 3. Find j* such that $\sum_{i=1}^{L} 1/x_{ij^*}$ is maximum for $j \in I_1$. That is for j* the total sample size is a maximum.

Now if j* satisfies all the constraints then X_j^* is feasible and the optimal solution is X_j^*. However if some of constraints $\frac{1}{N_i} \le X_{ij} \le 1$ are violated, we proceed as follows

Let $I = (i/\text{either } X_{ij}^* < \frac{1}{N_i} \text{ or } X_{ij}^* > 1)$

Fix $X_{ij} = \frac{1}{N_i}$ or $X_{ij} = 1$, as the case may be, for $i \in I$, and eliminate these strata from consideration. For the remaining strata find x_j for all $j \in I_1$ and repeat the process, using Result 2.4.1.

A general procedure along this line is possible that considers the intersection of some of the hyper plane, find the point of contact of the objective function with them, and proceeds until all the constraints are satisfied. However, this approach may turn out to be computationally not efficient if several intersections and their contact with the objective function have to be found.

2.5 Optimum Allocation (Chatterjee):

Chatterjee (1967) got an expression for the increase in variance of the mean for a stratified scheme, when a non-optimal allocation is used. The result is a generalization of Cochran (1963).

He also suggests a system of allocation based on measure of departure from the optimum for multivariate case.

For the cost of sampling let the linear cost function with no overhead cost $c = \sum c_i n_i$, let $n^0 = (n_1^0, n_2^0, \ldots, n_L^0)$

be the optimal allocation for a variate in a population with L-strata, w_i being the strata weight as the variance respectively.

Let $V(n^0)$ be the variance of the sample mean for the allocation of n^0-ignoring f.p.c, we have

$$V(n^0) = \frac{\left(\sum w_i \sigma_i \sqrt{c_i}\right)^2}{c}$$

Let n = $(n_1,\ldots\ldots,n_L)$ be another allocation for which cost is c, ignoring f.p.c, we get.

$$V(n) = \sum \frac{w_i^2 \sigma_i^2}{n_i}$$

Now we have after simplifications

$$\frac{V(n) - V(n^0)}{V\left(n^0\right)} = \frac{1}{c} \sum \frac{c_i \left(n_i^0 - n_i\right)^2}{n_i} \qquad \ldots\ldots (2.5.1)$$

It gives relative increase in the variance of an estimate of the sample mean when a non-optimal allocation is used. (2.5.1) is generalization of Cochran (1963).

If $\quad \max \frac{n_i^0 - n_i}{n_i} = g$

Then

$$\frac{V(n) - V\left(n^0\right)}{V\left(n^0\right)} \leq g^2 \qquad \ldots (2.5.2)$$

(2.5.2) given an upper bound in the variance. Chatterjee used (2.5.1) for devising a system of allocation in multi-variate stratified sampling. When several variates are under study, an allocation which may be optimum for one variate will not in general be optimum for another.

A compromise allocation may be chosen such that for each of the individual variates for the relative increase in variance from its optimum variance is as small as possible.

For a fixed c, let E_j denote the relative increases in the variance of the variate j when a non-optimal allocation is used. Then we have

$$\frac{V_j(n^o) - V_j(n)}{V_j(n^o)} = \frac{1}{n}\sum \frac{c_i (n^o_{ji} - n_i)^2}{n_i}$$

where n^o_{ji} denote the optimal allocation in the i^{th} stratum when the optimizing is done with respect to j^{th} variate, and n_i is the compromise allocation in the i^{th} stratum.

If there are k variates under enquiry, a system of allocation can be used which minimizes

$$E = \sum_{j=1}^{k} E_j \qquad \text{...... (2.5.3)}$$

Practically this means that we allocate the sample such that the total relative loss of precision is minimum.

This criterion is meaningful only if all the variates are of importance and we have to do the best that we can for a fixed budget our problem them is to minimize

$$E = \sum_j E_j = \frac{1}{c}\sum_j \sum_i \frac{c_i (n^o_{ji} - n_i)^2}{n_i}$$

Subject to. $c = \sum c_i n_i$

Using Lagrangian multipliers and simplifying, the compromise allocation is given by

$$n_i = \frac{k\sqrt{\sum_j n_{ji}^0}}{\sqrt{c_i}} \qquad \text{... (2.5.4)}$$

Where

$$k = \frac{c}{\sum_i \sqrt{c_i \sum n_{ji}^0}} \qquad \text{... (2.5.5)}$$

Again Chatterjee (1968) considered allocation problem in multivariate case where allocation is made in such a way that the sample estimates meet the stated levels of precision or tolerance at a minimum cost. Solution of the problem has been shown as a programming problem. The method considered by him is valid for any estimates (mean, totals, proportions) of the population and estimating method (Ratio and Regression etc.) for illustration.

If the problem of estimating the population mean is considered, with L variates. Let V_j^o be the specified variance tolerance for the mean of the jth variate and the cost of sampling be $c = \sum_{i=1}^{L} c_i n_i$, where c_i is the unite cost of sampling in the i^{th} stratum. The precision specification becomes.

$$V\left(\bar{y}_j\right) \le V_j^o \ , \ j = 1,2,........,p \qquad \text{....(2.5.6)}$$

is follows that the mean stratified sampling is

$$\Sigma \frac{w_i^2 \, s_{ji}^2}{n_i} \leq v_i^o + \Sigma \frac{w_i^2 \, s_{ji}^2}{N_i}$$

Where s_{ji}^2 is the variance in i^{th} stratum for the j^{th} variate.

If we put $x_i = \frac{1}{n_i}$, the allocation problem becomes.

Minimize $c = \Sigma c_i / x_i$

Subject to $\Sigma w_i^2 \, s_{ji}^2 \; x_i \leq v_j^o + \Sigma \frac{w_i^2 \, s_{ji}^2}{N_i}$

$$o \leq x_i \leq \frac{1}{N_i}$$

at least one unit is drawn from each stratum.

An algorithm has been developed by chatterjee to solve the above problem.

Starting with a non-optimal allocation we apply successive corrections to arrive at an optimal solution.

2.6 <u>Optimum Allocation (Khan)</u>:

S.U. Khan (1986) consider a survey in which one has to estimate 'p' characters of the individuals in 'k' different strata. It is assume that the strata boundaries are fixed in advance and the samples are chose independently and without replacement in the different strata. The sampling variance of an unbiased estimate of the mean of the jth character has the form

$$v^j = \sum_{i \in I} \left(\frac{1}{n_i} - \frac{1}{N_i} \right) v_{ij}, \; j \in J, \qquad \ldots\ldots \; (2.6.1)$$

Where

$I = 1, 2, \ldots, k$, $J = 1, 2, \ldots, p$, n_i are the sample allocations, N_i the strata sizes and V_{ij} are known constants. Let each individuals related in the sample be enumerated completely so that the cost function is linear. If c is the available budget and c_i is the enumeration cost per individual in the i^{th} stratum then the problem may be defined.

$$\sum_{i = I} c_i n_i \le c \qquad \ldots\ldots \; (2.6.2)$$

and

$$1 \le n_i \le N_i \; (i \in I) \qquad \ldots\ldots \; (2.6.3)$$

Since N_i are fixed the problem is equivalent to

$$\text{Minimize } v^j = \sum_{i \in I} \frac{v_{ij}}{n_i}, \; j \in J \qquad \ldots\ldots (2.6.4)$$

in the convex region defined by the linear constraints (2.6.2) and (2.6.3).

<u>Allocation for different characte</u> Consider the problem of minimizing (2.6.4) for $j = j'$, subject to the constraints (2.6.2). It has

been shown Khan, S.U. (1971) that an explicit expression for the solution is

$$\bar{n}_i^{j'} = \frac{(c_i v_{ij})^{1/2} c}{\left(c \sum_{i \in I} \left(c_i v_{ij} \right)^{1/2} \right)} \quad , i \in I \qquad \text{........ (2.6.5)}$$

If $n_i^{-j'}$ satisfy the conditions (2.6.3) then we take

$$n_i^{j'} = n_i^{-j'}, i \in I.$$

If some of the conditions in (2.6.3) are violated then define, I_1 and I_2 such that $n_i^{-j'} < 1$ for $i \in I_1$ and $n_i^{-j'} > N_i$ for $i \in I_2$.

Then the solution to (2.6.1) and (2.6.2) is given by

$$n_j^{-j'} = \left(c_i v_{ij} \right)^{1/2} \left(c - \sum_{t \in I_1} c_t - \sum_{t \in I_2} c_t N_t \right) \Big/ \left(c_i \sum_{t \in I} \left(c_i v_j \right)^{1/2} \right)$$

$$\text{for } i \in I - I_1 - I_2 \qquad \text{......} \quad (2.6.6)$$

$$n_i^{-j'} = 1 \quad \text{for } i \in I_1$$

$$n_i^{-j'} = N_i \quad \text{for } i \in I_2$$

We again test the conditions (2.6.3) for $n_i^{-j'}$ obtained in (2.6.6). If they are satisfied we put $n_i^{j'} = \bar{n}_i^{j'}$, otherwise repeat the process by defining new I_1 and I_2 until (2.5.3) hold for all $i \in I$.

In this ways we obtain 'p' different sets of allocations corresponding the various characters.

A Compromise Solution

The set $n_i^{j'}$, $i \in I$ of allocation obtained in (2.6.5) is best for jth character but may not be so far the others. Let the minimum values of v_j, $j \in J$, obtained by substituting the respective optimum $n_i^{j'}$ from (2.6.5) in (2.6.4) be m_j, $j \in J$. An ideal solution would have been the one at which $v^j = m_j$ for all $j \in J$. But such a solution is most likely not feasible.

A compromising solution will be chebyshev point i.e. a feasible point at a minimax distance to the ideal solution. To this end we have to solve the following convex programming problem.

Minimize $\quad W$

Subject to $\; v^j(n) - m_j < w, j \in J \qquad \qquad$ ....(2.6.7)

$$\sum_{i \in I} c_i n_i \le c,$$

and $\qquad 1 \le n_i \le N_i, \quad i \in I$

This problem reduces to the following convenient form by putting

$$n_i = \frac{1}{x_i}, i = 1, \ldots, k \text{ and } w = x_{k+1}$$

Minimize x_{k+1}

subject to $\sum\limits_{i \in I} v_{ij} x_i - n_j \leq x_{k+1}$, $\quad j \in J$ ……(2.6.8)

$$\sum_{i \in I} c_i x_i \leq c$$

and $\left(\frac{1}{N_i}\right) \leq x_i \leq 1, \quad i \in I.$

The minimum of x_{k+1} is obviously ≥ 0. Our problem is such that for $x_{k+1} = o$ than point in the region defined by

$$\sum_{i \in I} v_{ij} x_i \leq m_j, \quad j \in J \qquad \text{.....(2.6.9)}$$

Do not satisfy the non-linear constraint. Increase or decrease in x_{k+1} amount to a displacement in the linear constraint set. Our aim is to move the region defined by the linear constraints (2.6.9) through the changes in x_{k+1} such that this region just touches the feasible region defined by the non-linear constraint.

For this purpose we solve the following problem: (For solution method, see Khan, S.U. (1971).

Minimum $\sum\limits_{i \in I} c_i / x_i = F_o$

Subject to $\sum\limits_{i \in I} v_{ij} x_i \leq \left(n_j + x_{k+1}^{(1)}\right), j \in J$ ……(2.6.10)

And $\frac{1}{N_i} \leq x_i \leq 1, \; i \in I,$

where $x_{k+1}^{(1)}$ is some constant.

If $F_o - c \neq o$ then this implies that a feasible solution of the problem in (2.6.8) is not attained for this value of $x_{k+1}^{(1)}$. So we put.

$$x_{k+1}^{(2)} = x_{k+1}^{(1)} + \delta^{(1)},$$

where $\delta^{(1)} > 0 \text{ or } < 0$ according as F_o-c > or <o and then solve (2.6.10) with new value of x_{k+1}.

This process is continued with $x_{k+1}^{(1)} = x_{n+1}^{(i-1)} + \delta^{(i-I)}$, where $\delta^{(i)} = 2\,\delta^{(i-I)}$, until at r^{th} step, say, the sign of F_o-c change for the first time. Then for $(r+1)^{th}$ step we take $\delta^{(r)} = -\delta^{(r-1)}/2$ and $x_{k+1}^{(r+1)} = x_{k+1}^{(r)} + \delta^{(r)}$.

At further steps, say, i^{th} $\delta^{(r)} = \delta^{(r-1)}/2$ otherwise. The process terminates when $|F_o\text{-}c|$ is less than some pre-assigned small number. The values of n_i are obtained by the transformation

$$n_i = \frac{1}{x_i}, \quad i \in I.$$

Note that the values of n_i so obtained may be non-integral. An exact compromise integer's solution could be obtained by applying the branch and bound procedure.

Salkin (1975). Procedure is as follows:

(i) Arrange c_i, $i \in I$ such that $c(1) \geq c(2) \geq \ldots\ldots.. \geq c(k)$. Denote the corresponding n_i, $i \in I$.

Repeat the following procedure for j = 1, 2,..., k-1

(ii) At jth iteration we compute

$$|s_1| = \left| \sum_{i=1}^{j-1} c_{(1)} \bar{n}_{(i)} + c_{(j)} |n_{(j)}| + \sum_{i=j+1}^{k} c_{(i)} n_{(i)} - c \right|$$

and $$|s_2| = \left| \sum_{i=1}^{j-1} c_{(i)} \bar{n}_{(i)} + c_{(j)} |n_{(j)}| + 1 + \sum_{i=j+1}^{K} c_{(i)} n_{(i)} \right.$$

Where |x| represents the integral part of x,

Fix $|n_{(j)}| = \bar{n}_{(j)}$ if $|s_1| \leq ||s_2|$.

Otherwise fix $|n_{(j)}| + 1 = \bar{n}_j$.

(iii) For j = k, if s_2 >o then we should fix $|n_k| = \bar{n}_k$ even if $|s_1| \geq |s_2|$. This is done for maintaining $\sum_{i=1}^{k} c_{(i)} \bar{n}_{(i)} \leq c$.

$\bar{n}_i, i \in I$ Constitutes as approximate compromise integer solution for the allocation problem. An exact compromise integer solution could be obtained by applying the branch and bound procedure, (Salkin 1975).

Improvement in the compromise solution

After the above calculation one compute the objective vector $v^j(\underline{n})$, $j \in J$ by substituting the compromise solution $\underline{n}$ obtained from (2.6.10). Then compare $v^j(\underline{n})$ with 'j' for all $j \in J$. If all $v^j(\underline{n})$ are satisfactory, the improvements are not needed.

If some of the $v^j(\underline{n})$ are satisfactory and others are not then a certain amount of decrease must be accepted from m_j corresponding to the satisfactory $v^j(\underline{n})$, for allowing an improvement of the unsatisfactory ones in the next cycle. Let the index of the objective to be relaxed by j* (An information on the selection of the index j* can be obtained by performing a sensitivity analysis for the problem (2.6.8). Let m_j* be amount of decrease accepted. At the next cycle we solve the following problem corresponding to (2.6.10).

$$\text{Minimize } \sum_{i \in I} c_i / x_i$$

$$\text{subject to } \sum_{i \in I} v_{ij}\, x_i \le m_j + x_{k+1}^{(2)},\ j \ne j^* \qquad \text{(a)}$$

$$\sum_{i \in I} v_{ij^*}\, x_i \le \left(m_j^* - m_j^*\right) + x_{k+1}^{(2)} \qquad \text{(b)} \quad .....(2.6.11)$$

$$\text{and} \qquad \frac{1}{N_i} \le x_i \le 1, \quad i = I. \qquad \text{(c)}$$

The procedure used in (2.6.10), in also applied to solve the problem (2.6.11).

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CHAPTER III

Optimum Allocation Using Prior Information

3.1 Introduction:

Ericson (1965) used prior information for optimum allocation in stratified sampling with a single character under study. The case when 'p' population characteristics are to be estimated is also discussed under the assumption that the strata are sufficiently similar with respect to '$(p-1)$' characteristics.

Here we treat the problem when sampling is multipurpose and no assumption about the similarity of strata is made with respect to the different characters. A procedure for this problem is published in Ahsan and Khan (1977). The procedure consists of many phases. The sub-problem in the phases higher than two becomes tedious. Here we give another formulation of the problem which leads to a procedure in which the solution is easily obtained. This procedure is appear in Ahsan (1978). Ahsan and Khan (1982) considered the problem to minimize the total budgetary cost of the survey subject to the desire precisions assigned to the posterior variances of the population means when the sampling is multivariate.

3.2 Optimum Allocation without overhead cost:

Ahsan and Khan (1977) gave the following formulation of the problem of allocation for a stratified sample survey in which 'p' characters are defined on each element of the population. It is assumed that the prior information about the unknown within stratum means of the 'p' characters under study is available in terms of a multivariate normal distribution with known parameters.

Let the population of size N be divided into L non-overlapping strata of size N_h , $h=1,\ldots\ldots\ldots,L$. such that $\sum_{h=1}^{L} N_h = N$. Again let $W_h = \frac{N_h}{N}$, $h = 1,\ldots\ldots\ldots,L$, denotes the known proportion of population elements failing in the h^{th} stratum.

Let $\bar{y}_{hj}$, $j = 1,\ldots\ldots,p$, $h = 1,\ldots\ldots,L$, be the unknown within stratum mean for j^{th} characteristics in h^{th} stratum.

Let

$$W' = (W_1, W_2,\ldots\ldots\ldots, W_L)'$$

and
$$\bar{Y}_j' = (\bar{y}_{1j}, \bar{y}_{2j},\ldots\ldots\ldots \bar{y}_{Lj})'$$

where $(')$ stands for transpose.

Let overall population mean for the j^{th} characteristics is

$$\bar{y}_j = W'\bar{y}_j.$$

The n_h, $h = 1,\ldots\ldots\ldots,L$, denote the size of the independent sample drawn from the h^{th} stratum and let $\bar{x}_j = (\bar{x}_{1j}, \bar{x}_{2j},\ldots\ldots\ldots \bar{x}_{Lj})'$, $j = 1,\ldots\ldots\ldots,p$ be the vector of sample means for the j^{th} characteristics and s^2_{hj} be the known within stratum variance for the j^{th} characteristics in the h^{th} stratum.

It is assumed that $\bar{x}_j$ has a conditional L-variate normal distribution with mean $\bar{y}_j$ and variance-covariance matrix

$$M_j = D(s^2_{1j}/n_{1j},\ldots\ldots\ldots\ldots\ldots, s^2_{Lj}/n_{Lj}),$$

where $D(\ )$ represents a diagonal matrix of order $L \times L$ whose $(h,h)^{th}$ element is s^2_{hj}/n_{hj}.n_{hj} =number of individuals of the h^{th} stratum in the sample whose j^{th} characteristics has been measured, clearly $n_{hj} \le n_h$, $h = 1,\ldots\ldots,L$, $j = 1,\ldots\ldots,p$.

It is also assumed that the prior information about $\bar{y}_{hj}$ is available in terms of an L-variate normal distribution of $\bar{Y}_j$ with mean m_j and non-singular diagonal variance-covariance matrix A_j of order $k \times k$. Raiffa and Schlaifer (1961) showed that the posterior distribution of $\bar{y}_j$ for any given stratified sample and observed $\bar{x}_j$ in L-variate normal with mean

$$\bar{m}_j = \left|\bar{X}'_j M_j^{-1} + m'_j A_j^{-1}\right| \left|M_j^{-1} + A_j^{-1}\right|^{-1}.$$

Since $\bar{y}_j$ is a linear combination of $\bar{y}_{hj}$, $h = 1, \ldots\ldots, L$, it will have a univariate normal prior distribution with mean $W'm_j$ and variance $W'A_jW$ and a univariate normal posterior distribution with mean $W'm_j$ and $W'(M_j^{-1} + A_j^{-1})^{-1}W$

The total cost of the survey is

$$c = c_0 + \sum_{h=1}^{L} \sum_{j=1}^{P} c_{hj} n_{hj}$$

where c_0 = overhead cost of approaching the individual measurement and c_{hj} = per unit cost of measurement of j^{th} characteristics in h^{th} stratum.

The allocation problem can be stated as :

$$\text{Minimize} \quad c = \sum_{h=1}^{L} \sum_{j=1}^{P} c_{hj} n_{hj} \qquad \text{(a)}$$

Subject to

$$W'(M_j^{-1} + A_j^{-1})^{-1} W \le V_j, \qquad \text{(b)} \ldots\ldots\text{(3.2.1)}$$

$$\text{and} \quad n_{hj} \ge 0, \quad h = 1, \ldots\ldots, L, \quad j = 1, \ldots\ldots, P. \qquad \text{(c)}$$

where $V_j, j=1,\dots\dots\dots,P$ be the upper limits on the posterior variance of the estimate of $\overline{Y}_j$, fixed according to the requirements of precision.

The overhead cost c_0 is dropped from the objective function because it is independent of n_{hj}. Again

$$M_j^{-1} = D(n_{1j}/s_{1j}^2,\dots\dots\dots,n_{Lj}/s_{hj}^2)$$

and let

$$A_j^{-1} = D(a_{1j},\dots\dots\dots\dots,a_{Lj}).$$

Therefore,

$$(M_j^{-1}+A_j^{-1}) = D(a_{1j}+n_{1j}/s_{1j}^2,\dots\dots\dots,a_{Lj}+n_{Lj}/s_{Lj}^2)$$

and

$$(M_j^{-1}+A_j^{-1})^{-1} = D\left(\frac{1}{a_{1j}+n_{1j}/s_{1j}^2},\dots\dots\dots\dots\dots,\frac{1}{a_{Lj}+n_{Lj}/s_{Lj}^2}\right).$$

The constraints in 3.2.1(b) can thus be written as

$$\left(W_1,\dots\dots\dots\dots,W_L\right)\begin{pmatrix}\frac{1}{a_{1j}+n_{1j}/s_{1j}^2} & \dots\dots\dots\dots \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ \dots\dots\dots\dots & \frac{1}{a_{Lj}+n_{Lj}/s_{Lj}^2}\end{pmatrix}\begin{bmatrix}W_1\\ .\\ .\\ .\\ .\\ W_L\end{bmatrix} \le V_j.$$

Or $$\sum_{h=1}^{L}\frac{W_h^2}{a_{hj}+n_{hj}/S_{hj}^2} \le v_j \qquad j=1,2\dots\dots\dots\dots p$$

Or $$\sum_{h=1}^{L} W^2{}_h X_{hj} \le v_j \qquad j=1,2\dots\dots\dots p$$

Where $$X_{hj} = \frac{1}{a_{hj}+n_{hj}/S_{hj}^2} \qquad \dots..(3.2.2)$$

That is $$n_{hj} = \left(\frac{1}{X_{hj}} - a_{hj}\right) S_{hj}^2 \qquad \text{.......(3.2.3)}$$

Using transformation

$$C = \sum_{h=1}^{L} \sum_{j=1}^{p} \left(\frac{1}{X_{hj}} - a_{hj}\right) S_{hj}^2 C_{hj}$$

$$= \sum_{h=1}^{L} \sum_{j=1}^{p} \frac{C_{hj} S_{hj}^2}{X_{hj}} - \sum_{h=1}^{L} \sum_{j=1}^{p} a_{hj} S_{hj}^2 C_{hj}$$

the last term in the above expression is independent of n_{hj} therefore can be dropped from the objective function . Thus it is sufficient to

minimize $\sum_{h=1}^{L} \sum_{j=1}^{P} \frac{b_{hj}}{X_{hj}}$ only where $b_{hj} = C_{hj} S_{hj}^2$

Again under the same transformation the restriction $n_{hj} \geq 0$ became

$$X_{hj} \leq \frac{1}{a_{hj}} \qquad h = 1,2,.......,L, \quad j = 1,2,.........P$$

Finally, the allocation can be stated as :

Minimize $\sum_{h=1}^{L} \sum_{j=1}^{P} b_{hj}/x_{hj}$, (a)

Subject to

$$\sum_{h=1}^{L} W_h^2 x_{hj} \leq v_j, j = 1,............,P \qquad \text{(b)} \qquad \text{......(3.2.4)}$$

and $x_{hj} \leq \frac{1}{a_{hj}}, h = 1,........,L, j = 1,......,P..$ (c)

The problem (3.2.4) is a problem of non-linear programming problem in which the objective function is convex and the constraints are linear.

Ahsan and Khan gave a solution using Kuhn-Tucker theory as developed by Khan and Kokan (1967) by using n dimensional geometry.

3.2.1 The Solution:

In this section, another method for solving the problem (3.2.4) using K-T theory given by Kuhn-Tucker(1952) has been discussed. This method was developed by Ahsan & Khan (1977).

Let $X^{J(K)}$ denote the point of contract of the objective hyper sphere (3.2.4a) with the intersection of $k(\le p)$.

Hyper planes

$$\sum_{h=1}^{L} W_h^2 X_{hj} = V_j \qquad j \in J(K) \qquad \ldots.(3.2.5)$$

where $J(K)$ is the subset of the set of indices $(1,2,..........,p)$ such that $J(K)$ contains K indices out of p indices $(1,2,..........,p)$.

Let X^* denote the optimum solution to the problem (3.2.4) since the objective function (3.2.4a) is strictly convex for $X_{hj} > 0$ and the feasible region Γ given by the intersection of (3.2.4b) & (3.2.4c) is also convex, the value of X^* will be unique and it will be on the boundary of convex set Γ that is at X^*, some of the constraints will be satisfied with equality.

Again, if any of the X_{hj} is zero (3.2.4a) will become infinite. Thus in order to obtain X^* we must investigate the optimal solutions of the problems.

Minimize

$$\sum_{h=1}^{L}\sum_{j=1}^{p} b_{hj}/X_{hj}$$

subject to $\sum_{h=1}^{L} W_h^2 X_{hj} = V_j \qquad j \in J(K)$ .....(3.2.6)

and $X_{hj} > 0$, $h = 1,\ldots\ldots,L$ $j = 1,\ldots\ldots\ldots\ldots,P$, For all possible combination of J. By K-T theory two distinct solutions to (3.2.6) are

$$X_{hj} = \pm\frac{\Gamma b_{hj}}{\sum_{u\in J(k)} \lambda_{hu} W_h^2} \qquad \begin{array}{l} h = 1,\ldots\ldots\ldots,L \\ j = 1,\ldots\ldots\ldots,P \end{array} \qquad \text{.....(3.2.7)}$$

where λ_{hu} are Langrangian multipliers obtained by solving

$$V_r = \sum_{h=1}^{L} W_h^2 \frac{\sqrt{b_{hr}}}{\sum_{n\in J(k)} \lambda_{hu} W_h^2} \qquad r \in J(k) \qquad \text{.....(3.2.8)}$$

If $k = 1$, $X^{J(1)}$ will be the point of contact of the objective hyper surface with one of the constrained hyper plane say for $j = q$. Substituting the value of the λ_{hu} in (3.2.7) from (3.2.8), we have

$$X_{hj} = \frac{V_q\sqrt{p_{hj}}}{W_h \sum_{h=1}^{L} W_h\sqrt{p_{hj}}} \qquad \begin{array}{l} h = 1,\ldots\ldots\ldots,L \\ j = 1,\ldots\ldots\ldots,P \end{array} \qquad \text{......(3.2.9)}$$

In case $k > 1$, λ_{hu} in (3.2.8) can be evaluated by solving system of equations in (3.2.8) by the method given by Powell (1970).

3.2.2 The Procedure:

As starting point, we can take

$$\underline{X}^{\circ} = \left(X_1^0, X_2^0, \ldots\ldots\ldots\ldots X_p^0\right)$$

where $$X_j^0 = \left(\frac{V_j}{W_1^2}, \ldots\ldots\ldots\ldots, \frac{V_j}{W_L^2} \right)$$

i.e. $$X_{hj}^0 = \left(\frac{V_j}{W_h^2} \right), \quad h = 1,\ldots\ldots\ldots, L, p = 1,\ldots\ldots, P \qquad \ldots\ldots(3.2.10)$$

The solution $\underline{X}^0$ will violate all the constraints of the problem (3.2.4). The extent of violation for j^{th} constraints can be measured by the amount

$$d_j = \left(\sum_{h=1}^{L} W_h^2 X_{hj}^0 - V_j \right) \qquad j = 1,\ldots\ldots\ldots\ldots, p.$$

We can arrange these differences in descending order of magnitude. Let the corresponding indices are $j_1, j_2,\ldots\ldots, j_m, j_{m+1},\ldots\ldots\ldots\ldots, j_p$ that is

$$dm \geq d_{jm+1} \text{ for } m = 1,\ldots\ldots\ldots, p-1.$$

<u>Step-1</u> Obtain $\underline{X}^{(j)}$ by (3.2.9) for $j = j_1,\ldots\ldots, j_p$.

If $\underline{X}^{(j)}$ satisfies (3.2.4b) & (3.2.4c) for $j = j_u$.

$$X^* = X^{(j_u)}$$

If no such j_u exists then define

S(X) ={Constraints of (3.2.4b) which are violated by X}

Denote by $J^r(1)$, $r \in R_1$, say, the set of indices for which

$$S\left(XJ^r(1) \right) \neq \Phi$$

<u>Step-2</u> Using (3.2.7) & (3.2.8) $X^{J^r}(1) + S$.

Where $s \in S\left(\underline{X}J^r(1) \right)$ for all $X \in R_1$

If $S\left(XJ^r(1) + s \right) = \Phi$ and $S\left(\underline{X}J^r(1) \right) = s$

$$X^* = \underline{X}J^r(1) + s$$

Proceeding in this manner let one is unable to obtain j such that $\underline{X}\tilde{J} = \underline{X}^*$ up to $(m-1)^{th}$ step. Denote by $J^r(m-1)\ r \in R_{m-1}$, say, the set of those indices which $S\left(\underline{X}J^r(m-1)\right) \neq \Phi$.

Step m : Find $XJ^r(m-1)+s$ for $s \in S \qquad XJ^r(m-1) r \in R_{m-1}$

If $S\left(XJ^r(m-1)\right) + S = \varphi$ & $S\left(XJ^r(m-1)\right) = S$

$$X^* = XJ^r(m-1) + S$$

Ultimately if $\tilde{J}$ has exactly indices the solution will be obtain at m^{th} step.

Remark:- If any X_{hj} violates the restrictions of the problem (3.2.4), we will put the particular X_{hj} equal to its upper limit and solve the new problem. Consisting of $L(p-1)$ variables from beginning.

3.3 Another Approach

The objective function of the allocation problem given in (3.2.4) can be written as:

$$\sum_{h=1}^{L} \frac{b_{h1}}{x_{h1}} + \ldots\ldots\ldots\ldots + \sum_{h=1}^{L} \frac{b_{hj}}{x_{hj}} + \ldots\ldots\ldots\ldots + \sum_{h=1}^{L} \frac{b_{hp}}{x_{hp}}$$

In the above expression the j^{th} term contains only those variables, which are present in j^{th} constraint. Thus, we can separate the non-linear programming problem (3.2.6) as 'p' independent sub-problems:

$$\text{Minimize} \quad \sum_{h=1}^{L} \frac{b_{hj}}{x_{hj}},$$

$$\text{Subject to} \quad \sum_{h=1}^{L} W_h^2 x_{hj} \leq v_j \qquad \ldots..(3.3.1)$$

And $\qquad x_{hj} \le \frac{1}{a_{hj}}, h = 1,\dots\dots\dots\dots, L.$

3.3.1 <u>The Solution:</u> The lagrange form Φ for the problem (3.3.1) neglecting the restriction $x_{hj} \le \frac{1}{a_{hj}}, h = 1,\dots\dots\dots\dots, L$, is

$$\Phi = (x, \lambda) = \sum_{h=1}^{L} \frac{b_{hj}}{x_{hj}} + \lambda\left(\sum_{h=1}^{L} W_h^2 x_{hj} - v_j\right)$$

The Kuhn-Tucker (1952) conditions for the non-linear programming problem (3.3.1) are:

$$\frac{\partial \Phi}{\partial x_j} = 0 \text{ Where } x_j = (x_{1j}, x_{2j}, \dots\dots\dots\dots, x_{Lj})$$

$$\Rightarrow -\frac{bh_j}{x_{h_j}^2} + \lambda W_h^2 = 0$$

i.e. $\qquad x_{hj} = \sqrt{\frac{b_{hj}}{\lambda W_h^2}} \qquad \dots\dots\dots(3.3.2)$

Again, $\frac{\partial \Phi}{\partial \lambda} = \sum_{h=1}^{K} W_h^2 x_{hj} - v_j = 0 \qquad \dots\dots\dots(3.3.3)$

Eliminating λ from (3.3.2) and (3.3.3) we get

$$x_{hj} = \frac{v_j \sqrt{b_{hj}}}{W_h \sum_{h=1}^{L} W_h \sqrt{b_{hj}}.} \qquad \dots\dots\dots (3.3.4)$$

If x_{hj} given in (3.3.4) satisfies the restrictions in (3.2.) it will be optimum. If any x_{hj} violates the restrictions we can apply the given rule. If any x_{hj} violates the restrictions of the problem (3.3.4) we will put the particular x_{hj} equal to its upper limit and solve the new problem consisting of $L(p-1)$ variables from beginning.

3.4 Optimum Allocation with overheads cost:

Ahsan and Khan (1982) considered the problem again, where apart from the cost involved in enumerating the selected individuals in the sample, there is an overhead cost associated with each stratum and formulated as a problem of non-linear programming problem. The variances of the posterior distributions of the means of various characters are put to constraints and the total cost is minimized.

The main problem is broken into sub problems for each of which the objective function turns out to be convex.

When the number of sub problems happens to be large an approach has been indicated for obtaining an approximate solution by solving only a small number of sub problems.

Ahsan and khan (1982) also give a solution of the problem.

If the overhead cost c_0 is not independent of n_{hj} $h=1,\dots\dots,L,\ j=1,\dots\dots,p$, it could not be dropped from the objective function, of the minimization problem. Let 'c_h'denote the cost approaching an individual in the h^{th} stratum then the overhead cost

$$c_0=\sum_{h=1}^{L} c_h n_h \text{, where } n_h=\max_j . n_{hj} \quad \begin{matrix} h=1,\dots\dots,L \\ j=1,\dots\dots,p \end{matrix}.$$

The total cost of the survey in this situation will be

$$c=\sum_{h=1}^{L} c_h n_h+\sum_{h=1}^{L}\sum_{j=1}^{p} c_{hj} c_{hj}$$

Using the transformation (3.2.2) and neglecting the terms independent of n_{hj} the objective function of the allocation problem can be written as

$$\sum_{h=1}^{L} c_h \left[\underset{j}{\mathrm{Max}} \left(\frac{1}{X_{hj}} - a_{hj} \right) S_{hj}^2 \right] + \sum_{h=1}^{L} \sum_{j=1}^{p} \frac{b_{hj}}{X_{hj}} \qquad \text{.......(3.4.1)}$$

Let $H_j, j = 1,.....,p$ denote the set of those indices h for which $n_h = n_{hj}$ that is

$$H_j = \left[h / \underset{k}{\mathrm{Max}} \; n_{hk} = n_{hj} \right] k = 1,........,p \qquad \text{.......(3.4.2)}$$

Clearly H_j may be empty for one or more j and

$$\bigcup_{j=1}^{p} H_j = (1,2,.......,L).$$

Using the definition H_j given in (3.4.2) we can now state the optimum allocation problem with overhead cost as

$$\text{Minimize}\, c(\underline{X}) = \sum_{j=1}^{p} \sum_{h \in H_j} c_h \left[\left(\frac{1}{X_{hj}} - a_{hj} \right) S_{hj}^2 \right] + \sum_{h=1}^{L} \sum_{j=1}^{p} \frac{b_{hj}}{X_{hj}} \qquad \text{....(3.4.3)}$$

Subject to

$$\sum_{h=1}^{L} W_h^2 X_{hj} \le V_j \qquad j = 1,2,......,P$$

$$X_{hj} \le \frac{1}{a_{hj}} \qquad h = 1,2,.......,L$$

Where $\underline{X} = \left(X_{11},.........,X_{1p}, X_{21},.......,X_{2p},.............,X_{1L},...........,X_{Lp} \right).$

It can be seen that for $\underline{X} > 0$, $C(\underline{X})$ is convex.

If the sets $H_j, j = 1,\ldots\ldots,p$ are known the same technique as applied in section (3.3) of this chapter. It can be seen that this problem (3.4.3) is equivalent to following 'p' independent sub-problems.

$$\text{Minimize} \sum_{h=1}^{L} \frac{b_{hj}}{X_{hj}} + \sum_{h \in H_j} c_h \left(\frac{1}{X_{hj}} - a_{hj} \right) S_{hj}^2$$

$$\text{Subject to} \qquad \sum_{h=1}^{L} W_h^2 X_{hj} \le V_j$$

$$X_{hj} \le \frac{1}{a_{hj}}, \quad h = 1,\ldots\ldots\ldots\ldots,L.$$

In the above objective function, the term $-\sum c_h a_{hj} S_{hj}^2$ is independent of b_{hj} and therefore can be dropped from minimization.

Thus, the above problems can be written as

$$\text{Minimize} \sum_{h \notin H_j}^{L} \frac{b_{hj}}{X_{hj}} + \sum_{h \in H_j} \frac{b_{hj} + c_h}{X_{hj}} S_{hj}^2$$

$$\text{Or Minimize} \sum_{h=1}^{L} \frac{\bar{b}_{hj}}{X_{hj}} \qquad \text{(a)}$$

$$\text{Subject to } \sum W_h^2 X_{hj} \le V_j \qquad \text{(b)} \quad \ldots\ldots(3.4.4)$$

$$X_{hj} \le \frac{1}{a_{hj}} \quad h = 1,\ldots\ldots\ldots,L \qquad \text{(c)}$$

where

$$\bar{b}_{hj} = b_{hj} + c_n S_{hj}^2 \quad \text{if } h \in H_j$$

$$= b_{hj}, \text{ otherwise.}$$

3.4.1 **The Solution**:

The objective function (3.4.4a) is convex because $\bar{b}_{hj} > 0, h = 1,......., L, j = 1,..........., p$. Using a unique solution to the problem (3.4.4a) and (3.4.4b) for a fixed j is given by

$$X_{hj} = \frac{V_j \sqrt{\bar{b}_{hj}}}{W_h \sum_{h=1}^{L} W_h \sqrt{\bar{b}_{hj}}} \quad h = 1,........, L. \qquad ......(3.3.5)$$

Using transformation (3.3.3) then value of X_{hj} obtained from (3.4.5) the corresponding total cost of survey

$$c\left(H_1,..........., H_p\right) = \sum_{h=1}^{k} \sum_{j=1}^{P} c_{hj}\left(\frac{1}{X_{hj}} - a_{hj}\right) S_{hj}^2 + \sum_{j=1}^{p} \sum_{h \in H_j} c_h \left(\frac{1}{X_{hj}} - a_{hj}\right) S_{hj}^2$$

..... (3.4.6)

The set $H_j, j = 1,....., p$ are however unknown. There are p^L possible values of the group of sets $\left(H_1,.........., H_p\right)$. Our interest lies in finding the group of sets $\left(H_1^*,............., H_p^*\right)$ such that

$$c\left(H_1^*,............., H_p^*\right) = \min.\ c\left(H_1,................., H_p\right)$$

where the minimum has been taken over all possible values of the group of sets $\left(H_1,.........., H_p\right)$.

As there is no significance gain in precision by increasing the number of strata beyond 'b' for small values of 'p'. One can investigate all possible group of set $\left(H_1,.........., H_p\right)$.

For example: if $L=4$ and $p=3$. One has to select the minimum out of $3^4=81$ values of $c\left(H_1,............,H_p\right)$.

For every large values of 'p', Ahsan and Khan (1982) gave the following procedure for obtaining the approximation to the optimum solution in which only a small number of configuration of $\left(H_1,..........,H_p\right)$. are required to be tested.

In (3.4.6) the term

$$\sum_{j=1}^{p}\sum_{h\in H_j} c_h\left(\frac{1}{X_{hj}}-a_{hj}\right)S_{hj}^2$$

depend on $\left(H_1,..........,H_p\right)$.. The balancing factor for maximization of the above term over is S_{hj}^2.

Let $S_{hk}^2=\underset{j}{\text{Max}}\, S_{hj}^2$. The starting group of the sets $\left(H_1^{(1)},.....................H_p^{(1)}\right)$ is defined as

$$H_k^{(1)}=\left[h:\text{Max}\ S_{hj}^2=S_{hk}^2\right] \quad k=1,...............,p.$$

Denote the corresponding value of $C\left(H_1^{(1)},..........,H_P^{(1)}\right)$. by $c^{(1)}$. The other group of the sets $\left(H_1,..........,H_p\right)$. to be investigated for improvement in $c\left(H_1,..........,H_p\right)$. are those which are close to $H_j^{(1)}$, $j=1,........,p$ in the sense that $H_j, j=1,......,p$ consist of the indices h for which S_{hj}^2 are large.

Thus , $\left(H_1^{(2)},\ldots\ldots\ldots\ldots,H_p^{(2)}\right)=\left(H_1^{(1)},\ldots\ldots\ldots\ldots H_p^{(1)}\right)$ except that a row q^{th} whose indices belong to $H_t^{(1)}$, say. Now, belongs $H_t^{(2)}$, where 't'correspond to that column in the q^{th} row where next to the maximum over j of S_{qj}^2 is attained. After considering a convenient number of groups of the sets $\left(H_1,\ldots\ldots\ldots,H_p\right)$.. The approximate solution is naturally taken to be that one for which $c\left(H_1,\ldots\ldots\ldots,H_p\right)$ is minimum.

If some of the restrictions on X_{hj}, $h=1,\ldots\ldots,L\,,j=1,\ldots.,p$ given in (3.4.4c) are not satisfied. These particular values of X_{hj} are fixed equal to there upper limit and the problem is resolved for the remaining $(L-r)$ variables where r is the number of violated restrictions.

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CHAPTER IV

Use of Multivariate Information in Constructing the Estimates

4.1:- Introduction :

Consider a finite population of size N we are interested in estimating the population mean $\overline{Y}_N$ of a study variable Y, when information on an auxiliary variable X highly correlated with Y is readily available on all the units of the population. In sample surveys it is usual to make use of auxiliary information to increase the precision of estimators. It is well known that ratio and regression type estimators could be used for increased efficiency. Generally, it is the information on just one auxiliary variate that is used for purpose of sample selection or estimation. Quite often we possess information on several variates and it may be considered important to make use of the whole available information to improve the precision of at least some of the key items.

Olkin (1958) in his paper concerned with the extension of ratio estimates to the case where multi-auxiliary variables are used to increase precision. Raj (1965) proposed method of using information on several variats to achieve higher precision Mukherjee and Rao (1987) consider practical situation where information on two auxiliary variables related to the study variable is available at different levels and also study several estimators that arise naturally in this context and compare them under mean square error criterion and extend these results to the case when multiple auxiliary information is available.

4.2:-Multivariate ratio estimate

In Sample surveys precision in estimating the unknown mean $\overline{Y}$ of a finite population may be increased by using an auxiliary information variable X, which is correlated with Y and whose mean $\overline{X}$ is known. Olkin (1958) concerned the extension of ratio estimation to the case where multi-auxiliary variables are used to increase the precision. In this section Multi-variate ratio estimation presented by Olkin (1958) has been discussed.

In univariate case a simple random sample $(x_1, y_1), \ldots\ldots\ldots (x_n, y_n)$ from a finite population $(X_1, Y_1) \ldots\ldots\ldots, (X_N, Y_N)$ is observed. The mean $\bar{X}$ is known and $\bar{Y}$ is to be estimated. The estimator

$\tilde{Y} = \frac{\bar{y}}{\bar{x}}\bar{X} \equiv r\bar{X}$ is called ratio estimate for $\bar{Y}$. In general $\tilde{Y}$ is biased, and for large n approximation for $E(\tilde{Y})$ and $V(\tilde{Y})$ are given by

$$E(\tilde{y}) = \bar{Y} + \frac{N-n}{N}\frac{\bar{Y}}{n}(C_{XX} - C_{XY})$$

$$V(\tilde{y}) = \frac{N-n}{N}\frac{\bar{Y}^2}{n}(C_{XX} + C_{YY} - 2C_{XY})$$

In the multivariate extension we have the following model population.

$$Y_1, \ldots\ldots\ldots Y_N,\ \bar{Y} \text{ unknown}$$

$$X_{11}, \ldots\ldots\ldots X_{1N},\ \bar{X} \neq o \text{ known, } R_1 = \bar{Y}/\bar{X}_1$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$X_{P_1}, \ldots\ldots\ldots X_{P_N},\ \bar{X}_P \neq o \text{ known } R_P = \bar{Y}/\bar{X}_P$$

and $(P+1) \times (P+1)$ covariance matrix, S is known. The subscripts 0, 1, ..., P refer to Y, X_1, ..., X_P respectively, e.g. ρ_{02} is the correlation between y and X_2. Higher moments have superscripts referring to the variables and subscripts to the power e.g.

$$M_{12}^{ij} = \sum_K (X_{ik} - \bar{X}_i)(X_{jk} - \bar{X}_j)/N$$

$$M_{111}^{oij} = \sum_K (Y_k - \bar{Y})(X_{ik} - \bar{X}_i)(X_{jk} - \bar{X}_j)/N$$

Finally $S_{ij} = NM_{11}^{ij}/(N-1)$ denotes the covariance and $C_i = S_i/\overline{X}_i$ the coefficient of variation. Further, we denote

$$W_{12}^{ij} = M_{12}^{ij}/\overline{X}_i\,\overline{X}_j$$

A simple random sample $\left(Y_j, x_{ij}, \ldots, x_{pj}\right)$ where j=1, 2, ...n from the population is observed. The proposed ratio estimate of $\overline{Y}$ is

$$\tilde{y} = w_1 r_1 \overline{x}_1 + \ldots\ldots + w_p\, r_p\, \overline{X}_p \qquad \ldots.(4.2.1)$$

Where w = (w_1, w_2, ...w_p), $\sum w_i = 1$ is a weighting function and

$r_i = \overline{y}/\overline{x}_i$

As in univariate case $\tilde{Y}$ is biased in general and large sample. Approximation for the mean, variance and mean square error to $0(n^{-2})$ is given. Because of the complicated form of the terms of $0(n^{-2})$ and their dubious values only terms of $0(n^{-1})$ will be considered.

An optimal weight that minimizes the variance is also considered. From (4.2.1)

$$E(\tilde{y}) = \overline{Y} \sum w_i\, E\,(r_i/R_i) \qquad \ldots\ldots\ldots(4.2.2)$$

$$V(\tilde{y}) = \overline{Y}^2 \sum w_i\, w_j\, \mathrm{Cov}(r_i\, r_j) R_i\;\; R_j \qquad \ldots\ldots\ldots.(4.2.3)$$

In order to obtain approximation for $E(r_i)$ and Cov (r_i, r_j), we employ the usual delta method.

Let

$$E(\tilde{y})=\bar{Y}+\bar{Y}\frac{w\underline{b}}{n}+\bar{Y}\frac{wa}{n^2}+0(n^{-3}) \qquad \text{..........} \qquad (42.4)$$

$$V(\tilde{y})=\frac{\bar{Y}^2}{n}w(A+\frac{A}{n})\underline{w}+0(n^{-3}) \qquad \text{..........} \qquad (42.5)$$

Where vector $\underline{b}=(b_1,.........b_p)$ and $\underline{a}=(a_1,.........,a_p)$.

we further note that

$$b_i=\frac{N-n}{N}(C_i^2-\rho_{oi}C_o\ C_i)$$

$$a_{ij}=\frac{N-n}{N}(C_o^2-\rho_{oi}C_o\ C_i-\rho_{oj}C_o\ C_j+\rho_{ij}C_i\ C_j)$$

(For above computations see Sukhatme (1954)).

The criteria for optimality of the weight vector

w= (w_1, ... w_p) with $\sum w_i=1$ is to minimize $V(\tilde{y})$.

The obtain extremum, we make use of the generalized Cauchy inequality

$$(\underline{x}.y)^2\le(x\ \ M\underline{x})\left(y\ M^{-1}\ y\right) \qquad \text{...........} \qquad (4.2.6)$$

Where M is a symmetric positive definite matrix. The equality holds if $xM=\theta y$ where $\theta\neq0$ is a scalar.

Let $\underline{e}=(1,..........,1)$ and put $x=w,\ y=e$ and $M=A$. thus

$$1=(w\underline{e})^2\le(w\ A\underline{w})\ (eA^{-1}\underline{e})$$

Equality achieve iff $wA = \theta e$ or $w = \theta eA^{-1}$.By restriction $w\underline{e} = 1$ it follows that

$$\theta = \frac{1}{(eA^{-1}\underline{e})} \quad ,$$

and hence the optimum w is given by

$$w = \frac{eA^{-1}}{eA^{-1}\underline{e}} \qquad \text{..................} \qquad (42.7)$$

Substituting this value of w in (4.4) and (4.5)

$$E(\bar{y}) = \bar{Y} + \frac{\bar{Y}}{n}\frac{eA^{-1}\underline{b}}{eA^{-1}\underline{e}} \qquad \text{................} \qquad (42.8)$$

$$V(\bar{y}) = \frac{\bar{Y}^2}{n}\frac{1}{eA^{-1}\underline{e}} \qquad \text{..........} \qquad (42.9)$$

The bias is eliminated if $eA^{-1}\underline{b}$, this will hold if $b = 0$ i.e.

$$c_i = \rho_{oi}\, c_o \qquad \text{or} \qquad \bar{Y} = \bar{X}_i\, \rho_{oi}\, S_o/S_i\,, \quad (i = 1, 2, \ldots\ldots, P)$$

Which occurs when each regression taken individually passes through origin. The expression $eA^{-1}\underline{b} = 0$ does not hold expect for some exceptional cases.

4.3 Multivariate regression estimate:

Raj (1965) gave the expansions for the variance and unbiased estimator of the ratio estimate in the case of multistage design where sample of the first stage units is selected with PPS.

Let, there be P variates $x_1, x_2, \ldots\ldots\ldots\ldots, x_p$ for which the information is available on each unit in the population of size N. Let K_i be good estimate of R_i and R_i denotes the ratio of y on x_i. Then we use the following weighted difference estimator for estimating from a simple random sample of size 'n' the mean value of M of the character Y has been given by Raj (1965).

Where

$$\hat{M} = \sum_{i=1}^{P} w_i \, t_i \qquad \ldots\ldots\ldots\ldots \qquad (4.3.1)$$

$$t_i = \bar{y} - K_i (\bar{x}_i - \bar{X}_i) \qquad \ldots\ldots\ldots\ldots \qquad (4.3.2)$$

and $\bar{y}$, $\bar{x}_i$ are the sample means of y & x_i respectively and

$W_i (i = 1, \ldots\ldots\ldots p)$ are weights adding to unity.

The estimator $\hat{M}$ is unbiased estimate of M.

$$E(\hat{M}) = E\left(\sum_{i=1}^{P} w_i \, t_i \right)$$

$$= E\left(\sum_{i=1}^{P} w_i \left(\bar{Y} - K_i \left(\bar{x}_i - \bar{X}_i\right)\right)\right)$$

$$E(\hat{M}) = \sum_{i=1}^{P} w_i \, E(\bar{Y}) - \sum_{i=1}^{P} K_i \, w_i \, E(\bar{x}_i - \bar{X}_i)$$

$$= \sum_{i=1}^{P} w_i \; E(\overline{Y}) - o$$

$$= E(\overline{Y}) \sum_{i=1}^{P} w_i$$

$$= E(\overline{Y}), \qquad \sum_{i=1}^{P} w_i = 1$$

$= M$, because we are using SRS.

We have

$$E(\hat{M}) = M \quad \text{unbiased estimator of M}$$

variance of $M = V(M)$ and $V(M)$ is given by

$$V(M) = V\left(\sum_{j=1}^{P} w_i t_i\right) \qquad \ldots\ldots(4.3.3)$$

$$= \sum_{i=1}^{P} \sum_{j=1}^{P} w_i w_j \; Cov(t_i t_j)$$

Define S_{ij} as covariance between i and j. Let o, 1.....P corresponds to the variates y, X_1......, X_P respectively, we have

$$Cov\left(t_i, t_j\right) = E\left(\sum w_i\left(\overline{Y} - K_i\left(\overline{x}i - \overline{X}i\right) - M\right)\right) \times \left|\sum w_i\left(\overline{Y} - K_j\left(\overline{x}j - \overline{X}j\right) - M\right)\right|$$

$$= E\left|\overline{Y} - \sum w_i K_i\left(\overline{x}i - \overline{X}i\right) - M\right| \times \left|\overline{Y} - \sum w_i K_j\left(\overline{x}i - \overline{X}i\right) - M\right|$$

On simplification, we get

$$Cov(t_i, t_j) = \left(\frac{1}{n} - \frac{1}{N}\right) a_{ij} \qquad \ldots..(4.3.4)$$

Where

$$a_{ij} = S_{oo} - S_{ol} K_i - K_j S_{oj} - K_i K_j S_{ij}$$

From (4.3.3) and (4.3.4) we get

$$V(\hat{M}) = \frac{1}{n}\left(1-\frac{n}{N}\right)\Sigma w_i w_j a_{ij}$$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right) w\, A\, \underline{w} \qquad \text{......... (4.3.5)}$$

Where $A = (a_{ij})$ and $w' = (w_1 ...., w_p)$ .... (4.3.6)

Applying same technique as used by Olkin (1958), we get the optimum weights as

$$w = \frac{e\, A^{-1}}{e\, A^{-1}\, \underline{e}}$$

Where A is given in (4.3.6) and $e' = (1......,1)$.

Substitute w for w in (4.3.5), we get

$$V(M)_{opt} = \frac{1}{n}\left(1-\frac{n}{N}\right)\frac{e\, A^{-1}}{e\, A^{-1}\, \underline{e}} \qquad A\left(\frac{e\, A^{-1}}{e\, A^{-1}\, \underline{e}}\right)$$

$$= \frac{1}{n}\left(1-\frac{n}{N}\right) / e A^{-1}\, \underline{e}$$

4.4 <u>Sampling on more than two occasions</u>:

The general problem of replacement has been studied by Yates (1960) and Patterson (1950), with respect to both current estimates and estimates of change. When there are more than two occasion, the opportunities for a flexible use of the data are increased. Cochran (1963) gave the theory of sampling on many occasions. We will first summarize the same, and then extend it for multivariate case.

Let us suppose that the size n of the sample is same for all occasions. Let the simple random sampling be used and population variance be s^2 of of y_i same for all occasions, y_i is the estimation variable.

Notations:

$\bar{y}_{hu}$ =mean of the unmatched portion on occasion h with occasion (h-1).

$\bar{y}_{hm}$ = mean of matched portion on occasion h with occasion (h-1).

$\bar{y}_h$ = mean of the whole sample on occasion h.

On occasion h we may have some units of the sample that are matched with occasion $h-1$, some units are matching with both $(h-1)^{th}$ and $(h-2)^{th}$ occasions and so on.

On occasion h we may have some units of the sample matching with $(h-1)^{th}$ occasion, some unit matching with both $(h-1)^{th}$ and $(h-2)^{th}$ occasion and so on. To improve the current estimate we will try a multiple regression involving all matching to previous occasion

The two possible estimates are for unmatched, $\bar{y}'_{hu} = \bar{y}_{hu}$, and for matched $\bar{y}'_{hm} = \bar{y}_{hm} + b\left(\bar{y}_{h-1} - \bar{y}_{h-1,m}\right)$.

The variance of unmatched is given by

$$V\left(\bar{y}'_{hu}\right) = \frac{s^2}{u} \qquad \text{........(4.4.1)}$$

because simple random sampling is used.

Again the variance of matched portion can be obtained by using the following theorem of double sampling.

Theorem: If the first sample is of size 'n', the second sample is of size n and 1/n is negligible, the variance of the regression estimate in double sampling is given approximately by:

$$V(\bar{y}_{lr}) = \frac{s_y^2\left(1-\ell^2\right)}{n} + \frac{\ell^2 s_y^2}{n'}$$

In our present problem m crossponds to n and $\ell^2 S_y^2/n'$ corresponds to

$\ell^2 V(\bar{y}_{n-1})$ and $S_y^2 = s^2$.

Thus we get

$$V\left(\bar{y}'_{hm}\right) = \frac{s^2\left(1-\ell^2\right)}{m} + \ell^2 V(\bar{y}'_{h-1}) \qquad \text{........(4.4.2)}$$

Let m_h and n_h are numbers of matched and unmatched units respectively. Our problem is to determine the optimum values of m_h and u_n, which minimize variance of $\bar{y}'_h$. For this we first work out the variance of $\bar{y}'_h$.

From equation (4.4.1), we have

$$w_{hu} = \frac{1}{V(\bar{y}'_{hu})} = \frac{u}{s^2}$$

From equation (4.4.2), we have

$$w_{hm} = \frac{1}{V(\bar{y}'_{hm})} = \frac{1}{S^2\left(1-\ell^2\right)/m + \ell^2 V(\bar{y}'_{h-1})} \qquad \text{......(4.4.3)}$$

The best estimate of $\bar{y}'_h$ is

$$\bar{y}'_h = \theta_h \bar{y}'_{hu} + \left(1-\theta_h\right)\bar{y}'_{hm} \qquad \text{.....(4.4.4)}$$

Where

$$\theta_h = \frac{w_{hu}}{(w_{hu} + w_{hm})} \qquad \text{......(4.4.5)}$$

and

$$V(\bar{y}'_h) = V(\theta_h \bar{y}'_{hu} + (1-\theta_h)\bar{y}'_{hm}) \qquad \text{.......(4.4.6)}$$

putting the value of θ_h from (4.4.5) in (4.4.6) and simplifying, we get

$$V(\bar{y}'_h) = \frac{1}{w_{hu} + w_{hm}} = \frac{g_{hs^2}}{n} \qquad \text{.......(4.4.7)}$$

where g_h denotes of the variance of the occasion to that on the first occasion.

$$g_h = \frac{V(\bar{y}'h)}{\frac{s^2}{n}} \qquad \text{......(4.4.8)}$$

we get

$$\frac{n}{g_h} = \frac{s^2}{V(\bar{y}'h)} = \frac{s^2}{(w_{hu} + w_{hm})} = U_h + \frac{1}{\frac{(1-\rho^2)}{m_h} + \frac{\rho^2 g_{h-1}}{n}} \qquad \text{.......(4.4.9)}$$

Where $g_{h-1} = V(\bar{y}'h-1)/(S^2/n)$ by definition (4.4.8). But $m_h + u_h = n$ i.e. $u_h = n - m_h$.

Substituting this value of u_h in (4.4.9), we have

$$(n - m_h) + \frac{1}{\frac{(1-\rho^2)}{m_h} + \frac{\rho^2 g'h-1}{n}} = f(m) \quad \text{(say)} \qquad \text{.......(4.4.10)}$$

Cochran (1963) used the method of calculus for maximizing (4.4.10) which is equivalent to that of minimizing $V\left(\bar{y}'_h\right)$ because s^2 is constant.

Differentiating (4.4.10) w.r.t. m_h and equating to zero, we get

$$\frac{\partial f(m_h)}{m_h} = \frac{\sqrt{1-\rho^2}}{g_{h-1}\left(1+\sqrt{1-\rho^2}\right)} \qquad \text{.......(4.4.11)}$$

Now we will extend the above results for multivariate case.

Let there are P characters to be estimated. The correlation coefficient between matched portion of i^{th} character (i=1,...,p) on h^{th} and $(h-1)^{th}$ occasions.

We can set our problem as:

to find m_h which minimizes $V\left(\bar{y}'_h\right)$ such that

$V\left(\bar{y}'_{hi}\right) \le V_i, i = 1,..,p$

where V_i's are called the upper confidence bounds of $V\left(\bar{y}'_{hi}\right)$s. And are fixed according to required precision.

In other words we have to select m_h such that

$$\frac{s^2}{V\left(\bar{y}'_h\right)} \text{ is maximized}$$

Subject to constraints:

$$\frac{s_1^2}{V\left(\bar{y}'_{hi}\right)} \ge V_i, i = 1,..,p$$

and the restriction

$$u_h + m_h = n,$$

Now we have to show that the objective function is concave i.e.

$$\frac{\partial d^2}{\partial m_h^2} f(m_h) < 0$$

$$f(m_h) = (n - m_h) + \frac{1}{\frac{1-\rho^2}{m_h} + \frac{\rho^2 g_{h-1}}{n}}$$

$$= n - m_h + \frac{m_h}{a + b_{m_h}}$$

where a and b are positive constraints such that

$$a = 1 - \rho^2 \text{ and } b = \frac{\rho^2}{n} g_{h-1}$$

Again

$$f(m_h) = n - m_h + \frac{1}{b} - \frac{a}{b}\left(\frac{1}{a + b_{m_h}}\right).$$

Thus

$$\frac{\partial^2}{\partial m_h^2} f(m_h) = \frac{-2ab}{(a + bm_h)^\beta} < 0$$

$$\Rightarrow \frac{S^2}{V(\bar{y}_h)} \text{ is concave.}$$

The above problem can be solved by method of feasible directions.

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CHAPTER V

The Problem of Stratification in Multivariate Surveys

5.1 Introduction :

The problem of optimum stratification in multivariate surveys is that of cutting the strata boundaries so that the variance of the most important estimate is minimized while the other estimates do not cross the lower limits fixed for their precisions. since prior to the survey, the estimation variables are unknown, the stratification is being done by choosing the boundaries for an auxiliary variables which is closely related to the estimation variables. The auxiliary variables thus chosen will have a joint distribution with each of the estimation variables. Block (1958) has considered the problem, when the single estimation variable in the survey has a joint lognormal distribution with the auxiliary variable. In the following we consider the situation involving several estimation variables each having a joint distribution with the estimation variables and formulated the problem as a non-linear programming problem.

5.2 Formulation of the problem:

Consider p+1 estimation variable $y_1, y_2, \ldots .y_{p+1}$, and an auxiliary variable x, known as stratification variable, we have to divide the population into L strata so that the stratified sample, thus obtained gives the required optimum results. Assuming that each y_j (j=1,...., p+1) has, with x, a bivariate distribution with probability density function $f(x,y_j)$. For a sample of size n taken according to Neyman allocation from a stratified population, the variance of the sample mean is given by

$$V(x, y_j) = \frac{1}{n}\left(\sum_{h=1}^{L} p_{hj}\, s_{hj}\right)^2 \qquad \ldots..(5.2.1)$$

where x is a vector of population partition with components $x_o, x_1, \ldots. x_n$, such that

$$a = x_o \leq x_1 \leq x_2 \ldots. \leq x_n = b \qquad \ldots..(5.2.2)$$

a and b are known constraints. s^2_{hj} is the variance of the j^{th} estimation variable in the h^{th} stratum and

$$p_{hj} = \int_{x_{h-1}}^{x_h} \int_{-\infty}^{\infty} f\left(x_1, y_j\right) dy_j \, dx$$

It is assumed that the (p+1) th estimation variable is the most important one of the survey. Our problem consistent in finding a cut $x = (x_o, x_1, \ldots, x_n)$ which minimize the variance $V(x, y_{p+1})$ of (p+1) **th** estimation variable. Under the constraints.

$$\frac{1}{n}\left(\sum_{h=1}^{L} P_{hj} S_{hj}\right)^2 \leq bj, \qquad j = 1, \ldots, p \qquad \ldots\ldots(5.2.3)$$

and the restrictions (5.2.2), where b_j, $j=1,\ldots.,p$, are the specified upper limits upon the variance of y_j, $j=1,\ldots p$.

Since p_{hj} and S_{hj} are positive, the minimization of

$$V\left(x, y_{p+1}\right) = \frac{1}{n}\left(\sum_{h=1}^{L} p_{h(p+1)} S_{h(P+1)}\right)^2$$

Thus the problem of stratification can now be stated as the following non-linear programming problem.

Minimize $\qquad \phi(x, y_{p+1}) \qquad \ldots\ldots.(a)$

Subject to $\qquad \phi(x, y_j) \leq V_j, \; 1=1,\ldots.,p \qquad \ldots\ldots.(b)\ldots..(5.2.4)$

and $\qquad a = x_o \leq x_1 \leq x_2 \leq \ldots. \leq x_L = b \qquad \ldots\ldots.(c)$

where $V_j = \sqrt{h\, b_j}$, j=1,1,....p and

$$\phi(x,y_j) = \sum_{h=1}^{L} p_{hj}, s_{hj}, \; j=1,2,......p+1.$$

It can be seen that

$$\phi(x,y_j) = \sum_{h=1}^{L} \int_{x_{h-1}}^{x_h} \int_{-\infty}^{\infty} f\left(x,y_j\right) dx\, dy_j \int_{x_{h-1}}^{x_h} \int_{-\infty}^{\infty} y_j^2 f\left(x,y_j\right) dx\, dy_j$$

$$-\left[\int_{x_{h-1}}^{x_h} \int_{-\infty}^{\infty} y_j\, f\left(x,y_j\right) dx\, dy_j \right]^2]^{1/2} \qquad .....(5.2.5)$$

When the distribution of (x,y_j), j=1, ...p+1, is bivariate normal, the function $\phi(x,y_j)$ can be expressed as Khan (1968).

$$\phi(x,y_j) = \sum_{h=1}^{L}[\int_{u_{h-1}}^{u_h} e^{-u^2/2}\, du [\, (s_j^2\, (1-r_j^2)+u_j^2) \int_{u_{h-1}}^{u_h} e^{-u^2/2} du$$

$$+ r_j^2\, s_j^2 \int_{u_{h-1}}^{u_h} u^2\; e^{-u^2/2} du + 2u_j\, r_j\, s_j \int_{u_{h-1}}^{u_h} u e^{-u^2/2}\, du$$

$$-[r_j\, s_j \int_{u_{h-1}}^{u_h} u\, e^{-u^2/2}\, du + u_j \int_{u_{h-1}}^{u_h} e^{-u^2/2}\, du\,]^2\,]^{1/2}$$

$$.....(5.2.6)$$

Where $u = \frac{X-u_x}{s_x}$, u_x and u_j are the population means of x and y_j, j=1,....p+1 respectively, r_j is the coefficient of correlation between x and y_j, j = 1 ,....., p+1, and s_x^2 and s_y^2 are variance of x and y_j, j=1,2..., p+1.

Similarly, when x and y_j have a bivariate lognormal distribution ϕ (x,y_j) can be expressed as Block (1958).

$$\phi(x,y_j) = e^{(2h_j+s_j)^2} \sum_{h=1}^{L} e^{s_j^2} \int_{q_{h-1}}^{q_h} g(u)\,du \int_{q_{h-1}-2r_js_j}^{q_h-2r_js_j} g(u)du$$

$$-\left[\left(\int_{q_{h-1}-r_js_j}^{q_h-r_js_j} g(u)du\right)^2\right]^{1/2} \qquad \ldots\ldots..(5.2.7)$$

Where g (u) is the standard normal density, and

$$q_h = \frac{\log x_h - h_x}{s_x},\ h = o,\ 1,\ \ldots.,L \qquad \ldots\ldots..(5.2.8)$$

when x and y_j has a bivariate Pareto distribution, $f(x,y_j)$ has the form given below:

$$f(x,y_j) = \frac{p(p+1)(ab)^{p+1}}{(bx+ay_j-ab)^{p+2}}$$

$$x > a > o$$

$$y_j > b > o$$

$$= o \qquad x \le a,\ y_j \le b,\quad p > o, \qquad \ldots..(5.2.9)$$

Where a, b and p are parameters of the distribution.

ϕ (x,y_j) in this case can be given as

$$\phi(x,y_j) = \sum_{h=1}^{L}\Bigg[a^p\left(\frac{1}{X_{h-1}^p}-\frac{1}{X_h^p}\right)\Bigg\{1+b^2+b^2a^p\left(\frac{1}{X_{h-1}^p}-\frac{1}{X_h^p}\right)$$

$$+\frac{2ba^{p-1}}{p-1}\left(\frac{1}{X_{h-1}^{p-1}}-\frac{1}{X_h^{p-1}}\right)\Bigg\}+\frac{b^2a^{p-1}}{(p-1)^2}\left(\frac{1}{X_{h-1}^{p-1}}-\frac{1}{X_h^{p-1}}\right)$$

$$\left\{a^{p-1}\left(\frac{1}{X_{h-1}^{p-1}}-\frac{1}{X_h^{p-1}}\right)+2\right\}+\frac{2b^2a^{p-2}}{(p-1)(p-2)}\left(\frac{1}{X_{h-1}^{p-2}}-\frac{1}{X_h^{p-2}}\right)\Bigg]$$

5.3 Suggestions for the solution :

Considering the above objective function ϕ (x,y_j) and hence, also the constraints are not convex but they are smooth. If a suitable starting point is selected any algorithm for convex programming may converge to the solution. However in this section another approximate method has been suggested.

The function in the non-linear programming (5.2.4) are so complicated that it is hard even to test than for convexity and much effort is required in obtaining as absolute minimum by using the existing non-linear programming techniques.

A quadratic function is easily tested for convexity. Further the problem of minimizing a convex quadratic function with linear constraints are easily solved by existing convergent methods for quadratic programming (kunzi and Krelle (1962)). Also convergent algorithms are minimizing concave function with linear constraints (Tui (1964), Zwart (1974)).

A computational procedure for solving a non-linear programming problem by approximating its objective function by a quadratic function is discussed by Ahasn (1978b). The procedure used is that of 'Convex Chebyshev approximation' (Zukhovisky and Avdeyeva (1966)), with works well if the function to be approximated is smooth. The non-linearity's in the constraints of the problem can be, made linear by using the method devised by Miller (1963). If the approximate quadratic function turns out to be convex and the constraints of the problem are linear functions, then one can approximate the solution to the non-linear programming problem (5.4) by solving a quadratic programme. The computational experience suggests that a suitable choice of the starting point in the procedure may produce the desired convexity (or concavity) properties in the approximated quadratic function. Ahsan, Khan and Arshad (1983) also solve a numerical example to illustrate the details of the procedure.

5.4.Optimum allocation with several estimation variables:

The problem of allocating the sampling units to different strata so as to attain the desired precision for all the characters with minimum cost reduces to convex programming with linear constraints (Kokan and Khan (1967). In multivariate surveys where cost is fixed one would try to maximize the precision. However, an allocation advantageous to one character may produce unhappy results for the others. A unique objective function can be defined when a precise weight is known for each character in the survey [Roy 1971]. In the absence of the prior knowledge of relative weights a STEP method has been developed for linear programming with multiple objective functions. In this section, we discuss the method developed by Khan and Islam (1980) for the problem of allocating sampling units to different strata in multivariate stratified random sampling with fixed budget and in which objectives are convex functions.

Assume that the strata boundaries are fixed in advance and the samples are chosen independently in the different strata and without replacement.

Let there be p characters under study and k different strata. N_i be the size of ith stratum and n_i the sample size from the stratum. The sampling variance of an un-biased estimate of the mean of the jth character has the

$$V_j = \sum_{i \in I} \left(\frac{1}{n_i} \frac{1}{N_i} \right) V_{ij}, \; j \in J \qquad \text{........(5.4.1)}$$

where n_i are sample numbers, N_i the strata sizes and v_{ij} are known constants. If c is available budget and c_i is the enumeration cost per individual in the ith stratum. The cost of the survey assumed to be linear.

The problem is to minimizes (5.4.1)

$$\text{Subject to } \sum_{i \in I} c_i \, n_i \le c \qquad \text{........(5.4.2)}$$

$$\text{and} \qquad 1 \le n_i \le N_i \quad (i \in J) \qquad \text{........(5.4.3)}$$

Since N_i are given and fixed, the problem is equivalent to minimizing

$$\text{Min } V_j = \sum_{i \in I} (v_{ij} / n_i), \; i \in J \qquad \text{.........(5.13)}$$

In the feasible region defined by the constraints in (5.4.2) and (5.4.3).

solution for different characters they compute the values of the variances of different characters when an optimum allocation subject to one character is used

Let $n^{j'} = \left(n_1^{j'}, \ldots\ldots\ldots, n_k^{j'}\right)$ minimize $v^{j'}$ subject to (5.4.2) and (5.4.3).

It has been shown by [Khan,S.U. 1971] that explicit expression for the solution minimizing (5.4.1) subject to (5.4.2) is given by

$$\bar{n}_j^{j'} = \sqrt{(c_i v_{ij})}\ c/c_i \quad \sum_{t\in I}\sqrt{(c_i v_{ij})},\quad i\in I \qquad \ldots\ldots(5.4.5)$$

if $n_i^{j'}$ satisfies condition then we have

$$\text{If}\quad n_i^j = n_i^{-j'},\quad i\in I$$

After fixing n_i^j for $i \in I_1 + I_2$ the optimal solution to the problem (5.11), (5.12) and (5.13) is given by

$$n_i^j \sqrt{\left(c_i v_{ij'}\right)} \left(c - \sum_{t\in I_1} ct - \sum_{t\in I_2} c_t n_t\right) / c_1 \sum_{t\in I} \sqrt{\left(c_i v_{ij'}\right)}$$

for $i \in I - I_1 - I_2$

we repeat the process until (5.4.3) hold for all $i \in I$.

In this way allocation to different character can be obtain, these allocation may be non-integral.

An ideal solution would have been the one at which $v_j = m_j$ for all $j \in J$. But such a solution is most likely not feasible. Our next effort is to obtain a feasible point at a minim ax distance to the ideal solution.

A compromising solution will be Chebyshev point. To this end, the following convex programming problem has to be solved.

Minimize $\quad W$

Subject to $V^j(n) - m_j < W \quad , \quad j \in J$

$$\sum_{i \in I} c_i \, n_i < c \qquad \text{........(5.4.7)}$$

$$1 \leq n_i \leq N_i, \, i \in I$$

This problem reduces to the following convenient form by putting $n_i = \dfrac{1}{x_i}$ and $W = x_{k+1}$.

Minimize $\quad X_{k+1}$

Subject to $\quad \sum_{i \in I} v_{ij} \, x_j - m_j \leq x_{k+1}, j \in J$

$$\sum_{i \in I} c_i / x_i \leq c \qquad \text{......(5.4.8)}$$

$$(1/N_i) \leq x_i \leq 1, i \in I$$

The minimum of X_{k+1} is obviously greater than zero Our problem is such that for $X_{k+1} = o$ the point in the region defined by

$$\sum_{i \in I} v_{ij} \, x_i \leq m_j \, , \, j \in J \qquad \text{....(5.4.9)}$$

do not satisfy non-linear constraint. Our aim is to move the region defined by the linear constraints (5.4.9) through the changes in X_{k+1} such that these region just touches feasible region.

For this purpose we solve the following problem [for solution method see Khan, S.U. (1971)].

Minimum $\sum_{i \in I} c_i / x_i = F_o$

Subject to $\sum v_{ij} x_i \leq \left(m_j + X_{k+1}^{(1)}\right), j \in J$ .....(5.4.10)

and $1/N_i \leq x_i \leq 1, \quad i \in I,$

where $x_{k+1}^{(1)}$ is same constant.

If $F_o - c \neq o$ then, this implies that a feasible solution of the problem (5.17) is not attained for the this value of $x_{k+1}^{(1)}$. So we put

$$x_{k+1}^{(2)} = x_{k+1}^{(1)} + \delta^{(1)}$$

Where $\delta^{(1)}$ $> o$ or $< o$ according, as $F_o - c >$ or $<o$ and then solve (5.4.8) with new value of $x_{k+1}^{(1)}$. This process is continued with $x_{k+1}^{(1)} = x_{k+1}^{(i-1)} + \delta^{(i-1)}$,

where $\delta^{(1)} = 2\delta^{(i-1)}$ until, at rth step, say the sign of F_o-c changes for the first time. The process terminates when F_o-c is less than some pre-assigned small number. The values of n_i are obtained by the transformation.

$$n_i = 1/x_i, \quad i \in I$$

Note that the values of n_i so obtained by non-integral. An exact compromise integers solution could be obtained by applying the branch and bound procedure [See Salkin 1975].

After the calculation, one compute the objective vector $v^j(\underline{n})$, $j \in J$ by substituting the compromise solution $\underline{n}$ obtained from (5.4.10). Then compare $v^j(\underline{n})$ with m_j for all $j \in J$. If all $v^j(\underline{n})$ are satisfying the improvements are not needed. If some of $v^j(\underline{n})$ are satisfactory and other are not then a certain amount of decrease must be accepted from m_j corresponding to the satisfactory $v^j(\underline{n})$ for allowing an improvement of the unsatisfactory ones in the next cycle. Let the index of the objective to be relaxed by j. Let Δm_j be amount of decrease accepted at the next cycle. We solve the following problem corresponding to (5.4.10).

$$\text{Minimum} \quad \sum_{i \in I} c_i / x_i$$

$$\text{Subject to} \quad \sum_{i \in I} v_{ij}\, x_i \le m_j + x_{k+1}^{(2)}, \; j = j^*$$

$$\sum_{i \in I} v_{ij}^*\, x_i \le (m_j^* - \Delta m_j^*) + x_{k+1}^{(2)} \qquad \text{......(5.4.11)}$$

and $1/N_i \le x_i \le 1$, $i \in I$.

The procedure used in (5.4.10) will also apply in (5.4.11) to solve the problem.

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